Assignment 1

Steady convection-diffusion (1D)

1D convection-diffusion equation with constant coefficients and Dirichlet boundary conditions:

$$\begin{cases} au_x - \vartheta u_{xx} = s \quad x \in [0,1] \\ u(0) = u_0; u(1) = u_1 \end{cases}$$

Let us consider case when $\boldsymbol{s = 0, u_0 = 0, u_1 = 1}$. Solving this equation using Galerkin method with different parameters, the following results will be obtained.

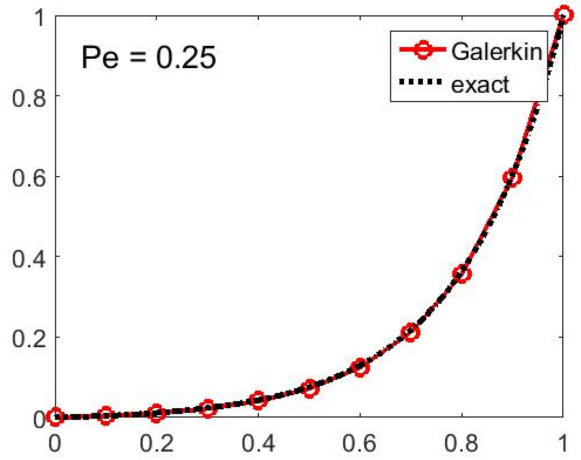

Fig.1. a = 1, ϑ = 0.2, 10 linear elements

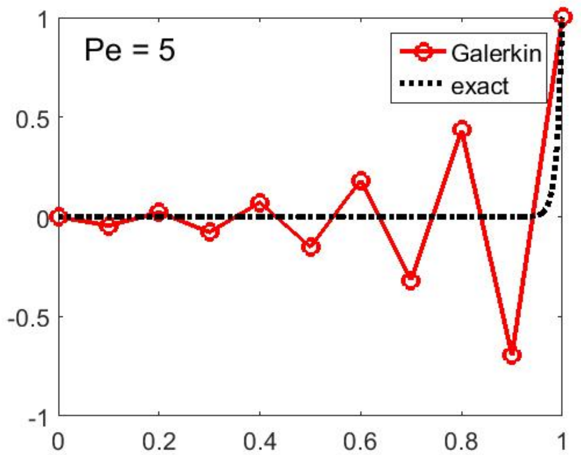

Fig. 2. a = 20, ϑ = 0.2, 10 linear elements

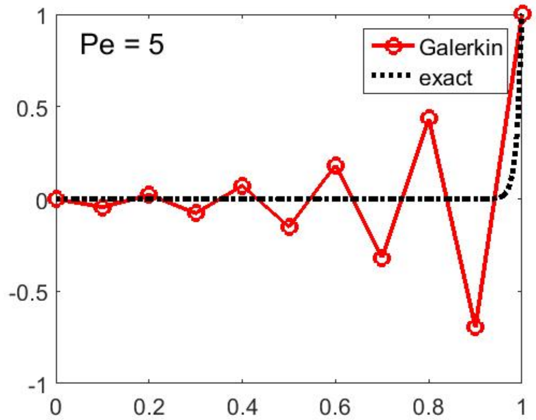

Fig. 3. a = 1, ϑ = 0.01, 10 linear elements

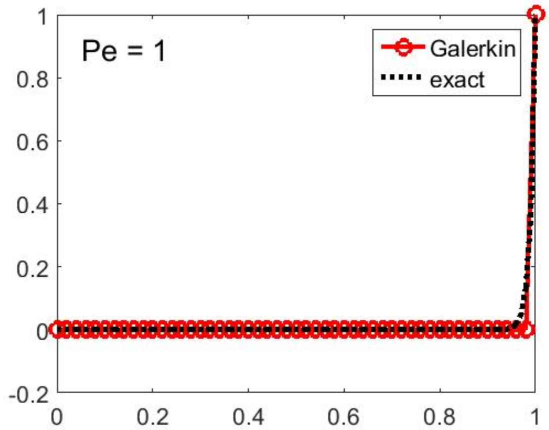

Fig. 4. a = 1, ϑ = 0.01, 50 linear elements

It is seen in figures 2 and 3 that Galerkin solution is corrupted by oscillations. It happens when Peclet number $\left(Pe = \frac{ah}{2\vartheta}\right)$ greater than 1. Galerkin methods has its best approximation property only when $Pe < 1$. It loses this property when the non-symmetric convection operator dominates the diffusion operator in the equation, and consequently oscillations appear. There are stabilization techniques which allow to stabilize approximation result when Peclet number is big.

Solving the equation with Streamline Upwind, Streamline Upwind Petrov-Galerkin (SUPG), Galerkin least squares (GLS) and Sub-grid scale (SGS) methods with a = 1, ϑ = 0.01,10 linear elements and optimal stabilization parameter $\tau = \frac{h}{2a}\left(\coth Pe - \frac{1}{Pe}\right)$, we receive the similar result for all four methods.

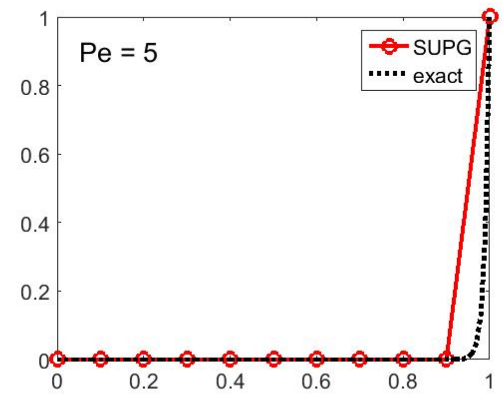

Fig. 5. Streamline Upwind, SUPG, GLS and SGS methods; a = 1, ϑ = 0.01, 10 linear elements, optimal τ

For these methods solution stable and close to exact solution when Peclet numbers are large. In this case stabilization parameter $\tau = 0.040005$. If stabilization parameter is chosen in different way, there will be oscillations or even solution can be completely different from exact one. Let us consider solution when $\tau = 0.01$ and $\tau = 1$.

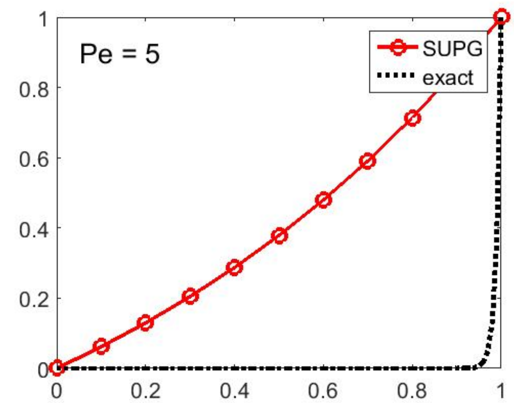

Fig. 6. SUPG, τ=1

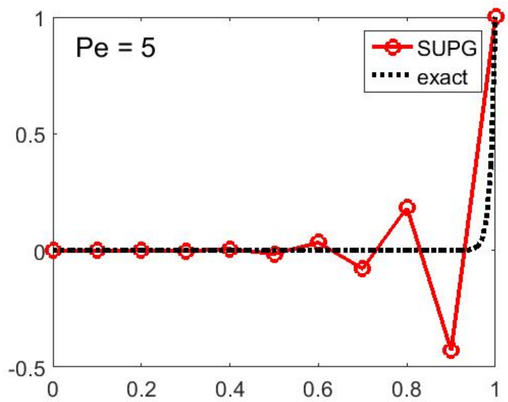

Fig. 7. SUPG, τ=0.01

It is seen from figures that for $\tau = 0.01$ some oscillations can be observed while for $\tau = 1$ the SUPG solution does not match with exact solution. The more stabilization parameter differs from optimal one, the more solutions differ from exact solution. Moreover, if stabilization parameter is equal to zero, methods behave as Galerkin method which is shown on the following graph.

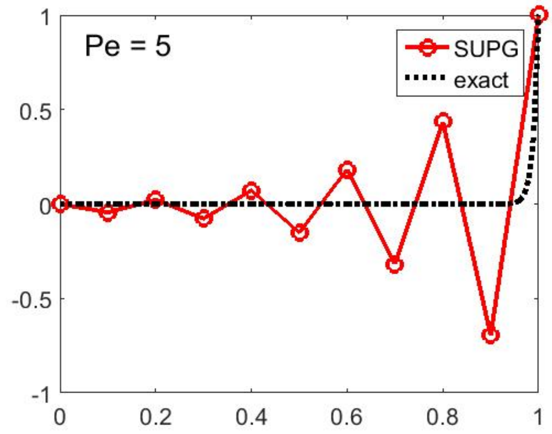

Fig.8. SUPG, τ=0

These examples reflect how it is important to choose correct stabilization parameter.

Let us consider the previous example solving it with SUPG method and quadratic elements.

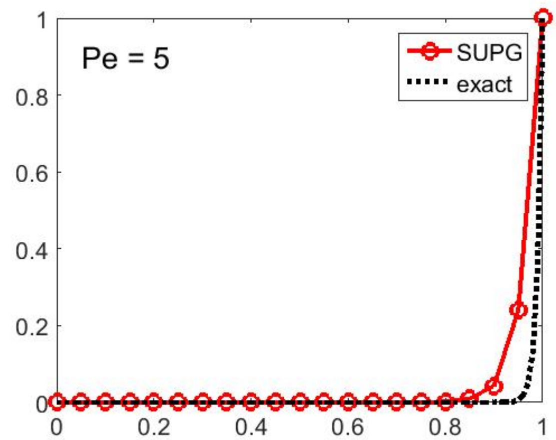

Fig.9. SUPG; a = 1, ϑ = 0.01, 10 quadratic elements, optimal τ

Comparing this figure with Fig. 5, we can see that use of quadratic elements instead of linear ones improves the solution which becomes closer to the exact solution.

Repeating all these experiments for case $\boldsymbol{s = \sin \pi x, u_0 = 0, u_1 = 1}$, we will see the similar behavior and results.

The Galerkin method gives good solution when Peclet number is less than 1 and oscillations otherwise.

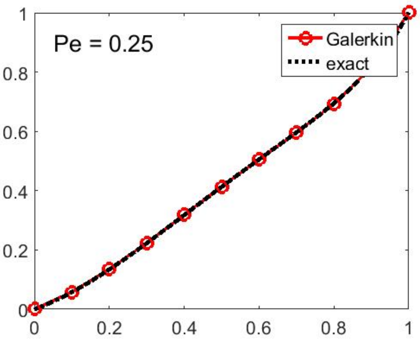

Fig. 10. a = 1, ϑ = 0.2, 10 linear elements

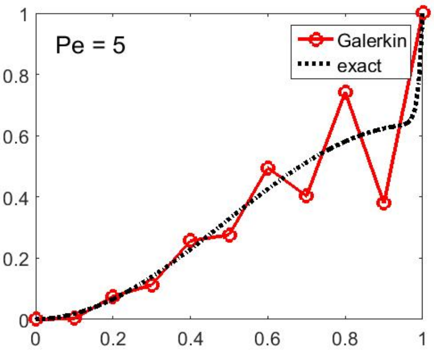

Fig.11. a = 1, ϑ = 0.01, 10 linear elements

Using stabilization techniques we can improve the solution if Peclet number is larger than 1. However, if stabilization parameter differs from optimal one, we receive oscillations and even wrong solution.

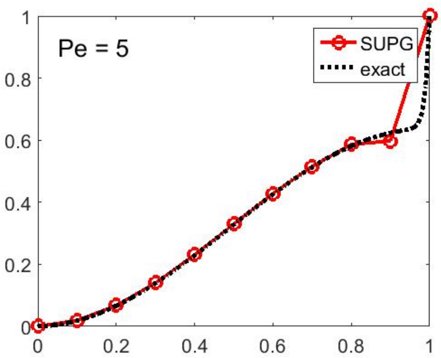

Fig. 12. optimal τ=0.040005

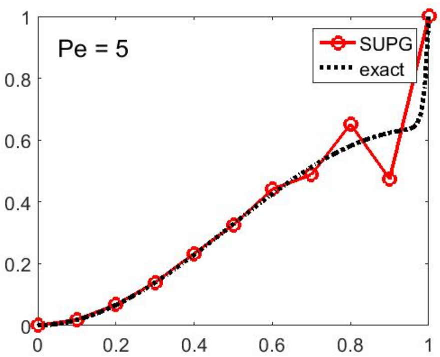

Fig. 13. τ=0.01

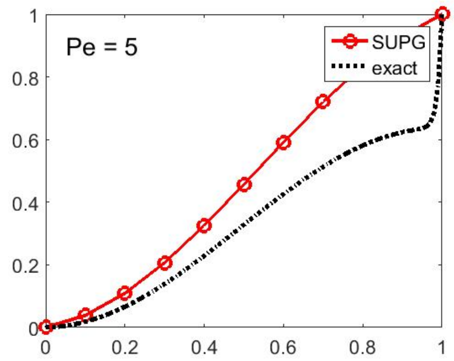

Fig. 14. τ = 1

However, we receive better solution if use quadratic elements instead of linear ones.

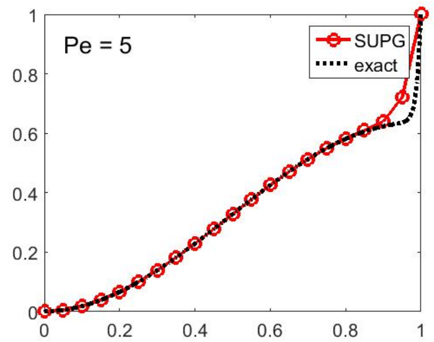

Fig. 15. Quadratic elements

Steady convection-diffusion-reaction (1D)

Now let us consider a 1D steady convection-diffusion-reaction problem

$$\begin{cases} au_x - \vartheta u_{xx} + \sigma u = s \quad x \in (0,1) \\ u(0) = u_0; u(1) = u_1 \end{cases}$$

Where $s = 0, u_0 = 0, u_1 = 1, a = 1, \vartheta = 0.01, \sigma = 20$. Solving this problem with different methods with 10 linear elements, we will receive following results. As we do not know exact solution for current problem, there are only graphs of methods. Stabilization parameter is counted as

$$\tau = \frac{h}{2a}\left(1 + \frac{9}{Pe^2} + \left(\frac{h}{2a}\sigma\right)^2\right)^{-1/2}.$$

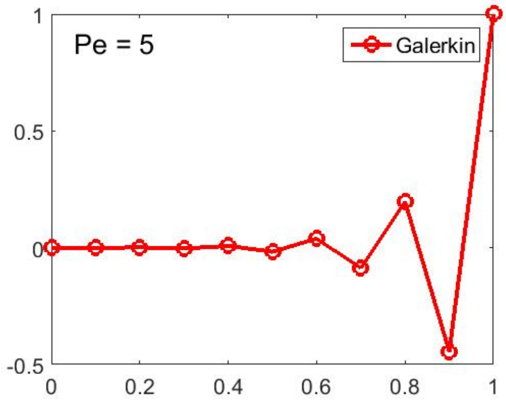

Fig. 16. Galerkin method

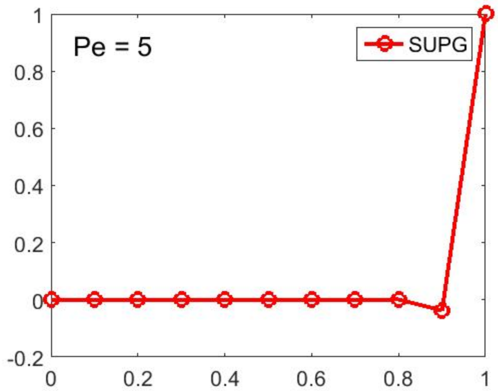

Fig. 17. SUPG method, optimal τ = 0.032547

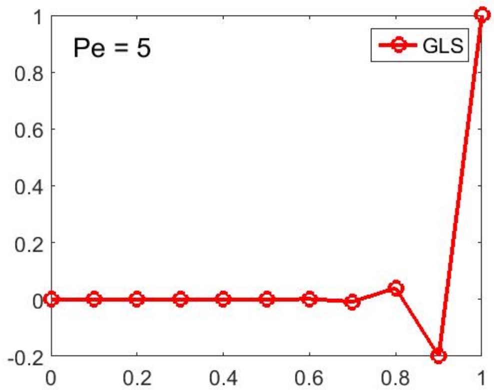

Fig. 18. GLS method, optimal τ = 0.032547

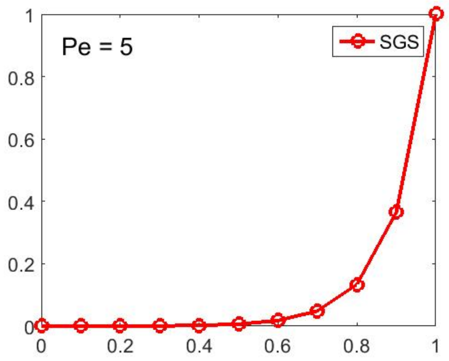

Fig. 19. SGS method, optimal τ = 0.032547

As we can see from Fig. 16, Galerkin method has oscillations because Peclet number is greater than 1. SUPG and GLS give more accurate solution however with slight fluctuations. The qualitative influence of each term of stabilization term $\mathcal{P}(w) = \mathcal{L}(w) = a \cdot \nabla w - \nabla \cdot (\vartheta \nabla w) + \sigma w$ for GLS method where $a \cdot \nabla w$ corresponds to SUPG method and σw is a Galerkin weighting. For linear elements and a constant positive reaction, GLS and SUPG with Galerkin weighted 1 + σ τ times more. This means that the instabilities introduced by Galerkin are little more amplified in GLS to compare with SUPG. This instability is overcome in SGS method. As $\mathcal{P}(w) = -\mathcal{L}^*(w) = a \cdot \nabla w + \nabla \cdot (\vartheta \nabla w) - \sigma w$, in this case Galerkin term is weighted by 1 - σ τ and thus has less influence than SUPG.

To improve results of Galerkin method, we can improve mesh. For example, it is enough to use more elements in a mesh. Increasing the number of linear elements to 70, we receive a Pe = 0.71429. In this case Galerkin method provides solution which is close to exact one.

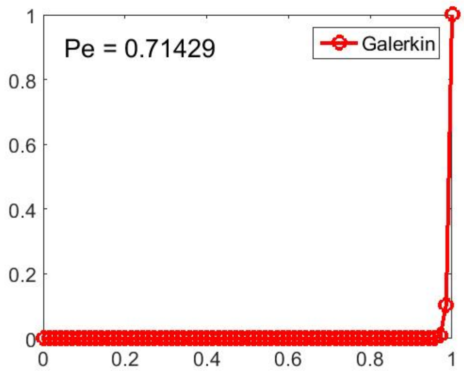

Fig. 20. Galerkin method, 70 linear elements

Steady convection-diffusion-reaction (2D)

Let us consider 2D steady convection-diffusion equation in domain [0,1]x[0,1] where $\|a\| = 1, \vartheta = 0.0001$. Let us impose zero Dirichlet boundary conditions on the outlet boundary condition. The solution now involves a thin boundary layer.

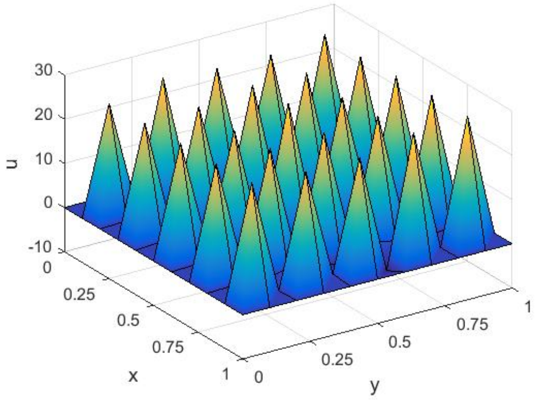

Fig.1. Galerkin method

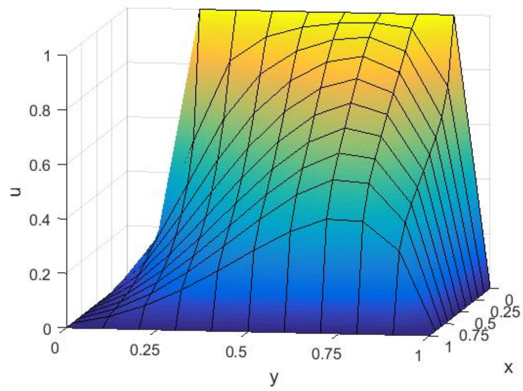

Fig. 2. Artificial diffusion method

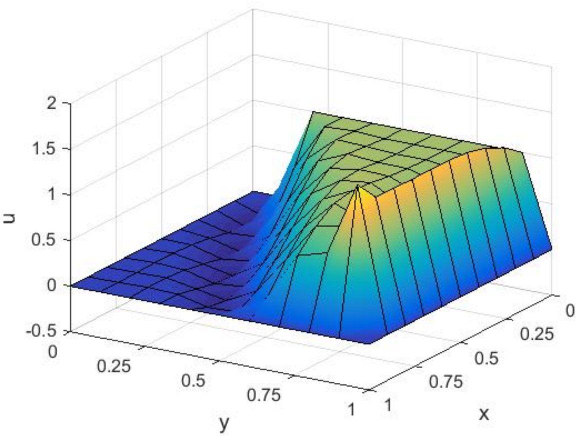

Fig. 3. SUPG method

The Peclet number is equal to 5 and as it greater than 1, Galerkin method have big oscillations. Its solution does not have any similarity with exact solution. Meanwhile the case when Neumann boundary conditions were imposed, although Galerkin solution had some oscillations, it was closer to exact solution. Artificial diffusion and SUPG methods provide better results. Artificial diffusion method introduces excessive numerical diffusion.

Now let us consider convection-reaction dominated problem with zero Dirichlet boundary conditions on the boundary with $\|a\| = \frac{1}{2}, \vartheta = 0.0001, \sigma = 1.$

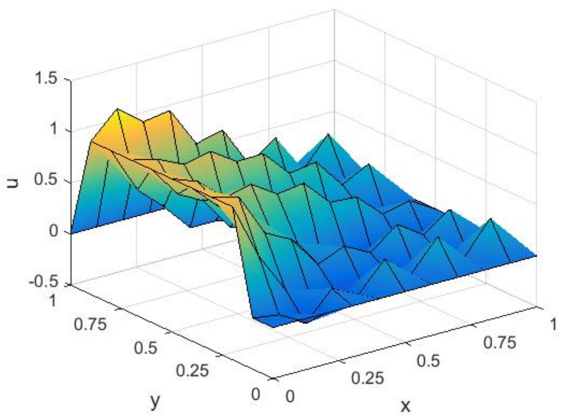

Fig. 4. Galerkin method

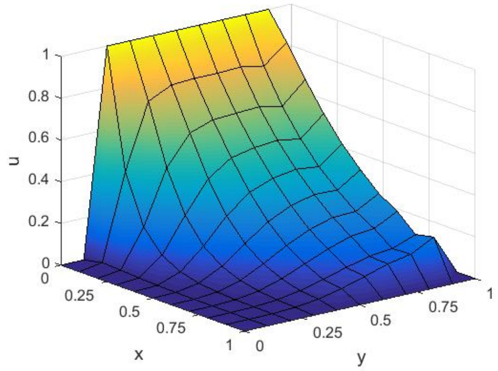

Fig. 5. Artificial diffusion method, optimal $\tau = 0.0084877$

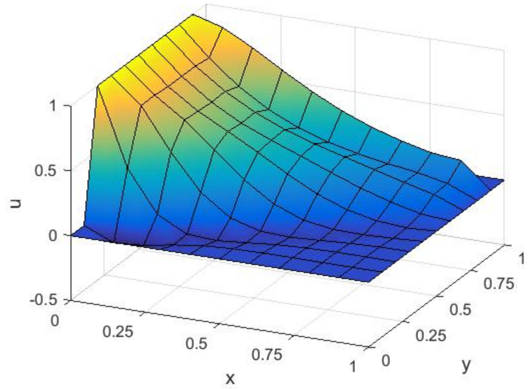

Fig. 6. SUPG method, optimal $\tau = 0.099497$

As Peclet number is still very high, Galerkin method demonstrates wild oscillations. Artificial diffusion and SUPG methods provide the similar results which are close to exact solution. There are only some oscillations near the boundary layers.

Now let us consider reaction dominated problem with zero Dirichlet boundary conditions on the boundary with $\|a\| = 0.001, \vartheta = 0.0001, \sigma = 1$.

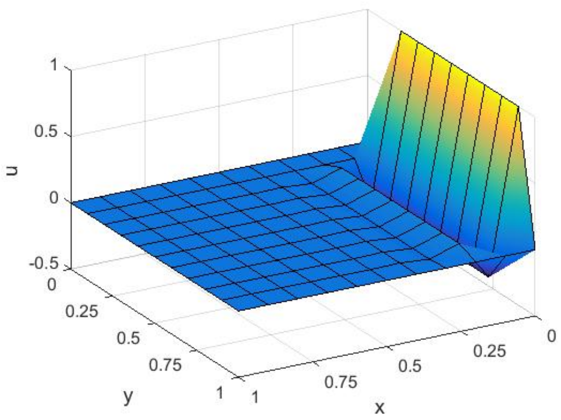

Fig. 7. Galerkin method

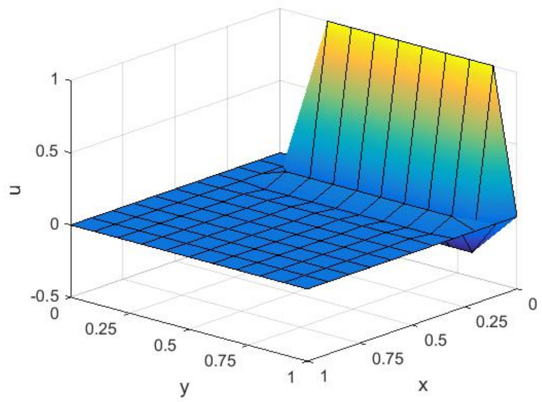

Fig. 8. Artificial diffusion method, optimal τ=8.2479e-12

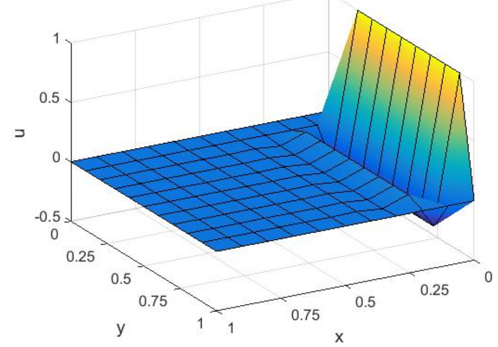

Fig. 9. SUPG method, optimal τ= 0.99268

In this case Pe=0.5 which leads that Galerkin method provides solution close to exact one. All methods show similar results with some fluctuations near the boundary layers.

Unsteady transient convective problem (1D)

Solving the transient convection-diffusion dominated equation

$$u_t + a u_x + \vartheta u_{xx} = 0$$

With BC:

$$u(x,0) = \frac{5}{7}\left(- \left(\frac{x - x_0}{L} \right)^2 \right)$$

Analytical solution:

$$u(x,t) = \frac{5}{7\sigma}\left(- \left(\frac{x - x_0 - at}{\sigma L} \right)^2 \right)$$

Where

$$\sigma = \sqrt{1 + \frac{4\nu t}{L^2}}, \qquad x_0 = \frac{2}{15}, \quad L = \frac{7\sqrt{2}}{300}$$

Using Crank-Nicolson method for time and a Galerkin for space with different Courant and Peclet numbers, receive:

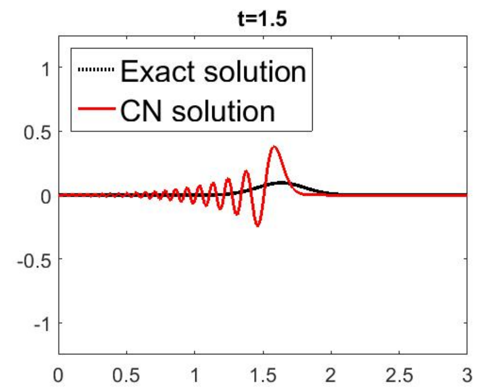

Fig. 1. C=0.5, Pe=0.0625, diffusion parameter $\vartheta = 0.01$

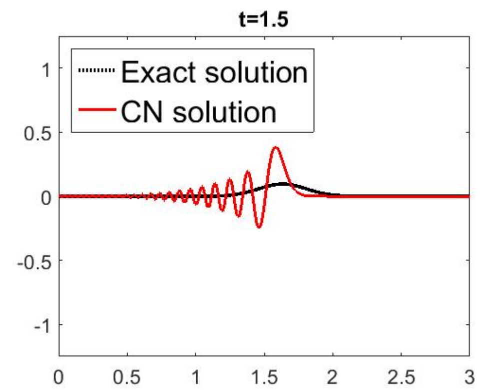

Fig. 2. C=1.0417, Pe=0.3, diffusion parameter $\vartheta = 0.01$

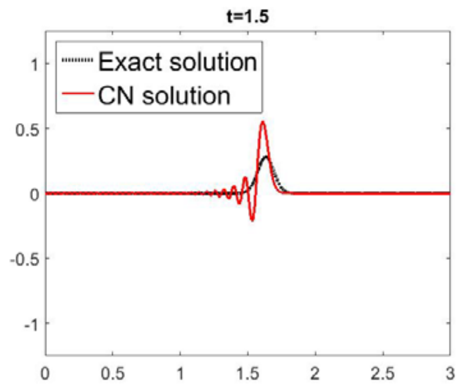

Fig. 10.C=0.5, Pe=6.25, diffusion parameter ϑ=0.001

As Crank-Nicolson method is a particular θ-method whith $\theta = 1/2$, it is unconditionally stable. Stability of method does not depend on Courant number for $\theta \leq 1/2$. However we can observe oscillations which are caused by Galerkin method which is conditionally stable and its stability depends on Peclet number.

Let us consider transient problem

$$u_t + au_x = 0$$

Solving this equation with Leap-Frog method, receive following results.

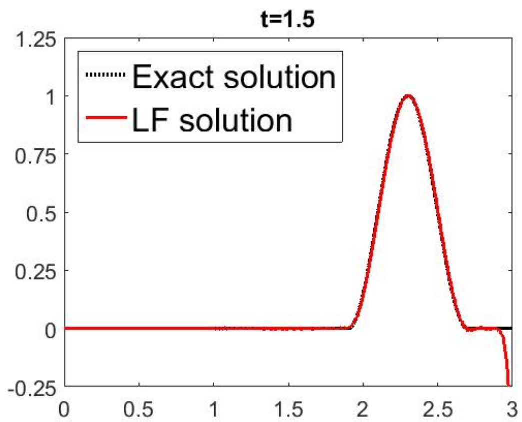

Fig. 11. C=0.5

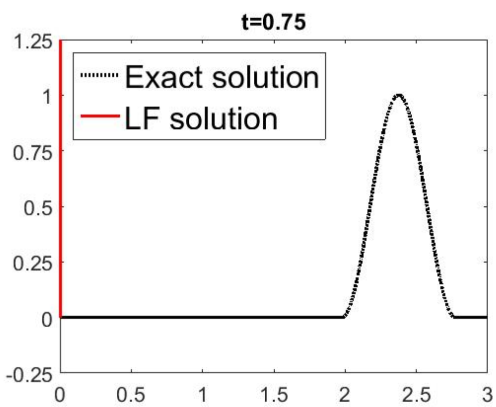

Fig. 12. C= 1.05

1D Leap-frog method provides stable solution when $C^2 < 1$. As we can see from the Fig. 12 where Courant number is 1.05, Leap-Frog solution is wildly unstable. However in the Fig. 11, where C=0.5, Leap-Frog solution coincides exact solution.