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$$\begin{cases} -\nabla\cdot\bar{\bar{\sigma}} = \vec{g} \\ \nabla\cdot\vec{v} = 0 \\ \vec{v} = \vec{v}_D \quad \text{on } \Gamma_D \\ n\cdot\bar{\bar{\sigma}} = \hat{t} \quad \text{on } \Gamma_N \end{cases}$$

$\longrightarrow$ traction can be naturally imposed

with

$$\bar{\bar{\sigma}} = -p\mathbb{I} + \tau$$

we move from strong form to weak form:

we first multiply by weight function

$$\begin{cases} -\int_\Omega \vec{w}\,\nabla\cdot\bar{\bar{\sigma}}\,d\Omega = \int_\Omega \vec{w}\,\vec{g}\,d\Omega \\ \int_\Omega q\nabla\cdot\vec{v}\,d\Omega = 0 \end{cases}$$

Now we integrate by parts the first term:

we know that:

$$\nabla\cdot(\vec{w}\cdot\bar{\bar{\sigma}}) = \nabla\vec{w} : \bar{\bar{\sigma}} + \vec{w}\cdot(\nabla\cdot\bar{\bar{\sigma}})$$

then:

$$\int \vec{w}\,\nabla\cdot\bar{\bar{\sigma}}\,d\Omega = \underbrace{\int \nabla\cdot(\vec{w}\cdot\bar{\bar{\sigma}})\,d\Omega}_{\text{divergence theorem}} - \int \nabla\vec{w} : \bar{\bar{\sigma}}\,d\Omega$$

$$\int_\Omega \nabla\cdot(\vec{w}\,\bar{\bar{\sigma}})\,d\Omega = \int_{\partial\Omega} \vec{w}\,\bar{\bar{\sigma}}\,\hat{n}\,d\Gamma$$

then

$$\underset{\Omega}{\int} \vec{w}\, \nabla \cdot \bar{\sigma}\, d\Omega = \underbrace{\underset{\partial\Omega}{\int} \vec{w}\, \vec{\vec{\sigma}} \cdot \hat{n}\, d\Gamma}_{\text{we focus on that: given that test functions take value equal to } 0 \text{ at dirichlet B.C.}} - \underset{\Omega}{\int} \nabla \vec{w} : \vec{\vec{\sigma}}\, d\Omega$$

$$\underset{\partial\Omega}{\int} \vec{w}\, \vec{\vec{\sigma}} \cdot \hat{n}\, d\Gamma = \underset{\Gamma_D}{\int} \underbrace{\vec{w}}_{0}\, \vec{\vec{\sigma}} \cdot \hat{n}\, d\Gamma + \underset{\Gamma_N}{\int} \vec{w}\, \vec{\vec{\sigma}} \cdot \hat{n}\, d\Gamma$$

then

$$\boxed{\underset{\Omega}{\int} \vec{w}\, \nabla \cdot \vec{\vec{\sigma}}\, d\Omega = \underset{\Gamma_N}{\int} \vec{w}\, \vec{\vec{\sigma}} \cdot \hat{n}\, d\Gamma - \underset{\Omega}{\int} \nabla \vec{w} : \vec{\vec{\sigma}}\, d\Omega}$$

and knowing that $\hat{n} \cdot \vec{\vec{\sigma}} = \hat{t}$

our first equation reads:

$$\boxed{\begin{cases} \underset{\Omega}{\int} \nabla \vec{w} : \vec{\vec{\sigma}}\, d\Omega = \underset{\Omega}{\int} \vec{w} \cdot \vec{g}\, d\Omega + \underset{\Gamma_N}{\int} \vec{w} \cdot \hat{t}\, d\Gamma \\ \underset{\Omega}{\int} q\, \nabla \cdot \vec{v}\, d\Omega = 0 \end{cases}}$$

this is the weak form!!

with that, now we can write $\vec{\vec{\sigma}} = -p\mathbb{I} + 2\mu \nabla^{s} \vec{v} =$

$$\vec{\vec{\sigma}} = -p\mathbb{I} + 2\mu \left(\frac{\nabla \vec{v} + \nabla \vec{v}^{T}}{2} \right)$$

then the first term:

$$\underset{\Omega}{\int} \nabla \vec{w} : \vec{\vec{\sigma}}\, d\Omega = -\underset{\Omega}{\int} p\, \nabla \cdot \vec{w}\, d\Omega + \underset{\Omega}{\int} \nabla \vec{w} : (2\nabla \vec{v})\, d\Omega$$

Then the weak form reads:

$$\begin{cases} \int_\Omega \nabla\vec{w} : (\nu \nabla\vec{v})\, d\Omega - \int_\Omega (\nabla\cdot\vec{w})\, p\, d\Omega = \int_\Omega \vec{w}\cdot\vec{f}\, d\Omega + \int_{\Gamma_N} \vec{w}\cdot\hat{t}\, d\Gamma \\ \text{and} \quad \int q\, \nabla\cdot\vec{v}\, d\Omega = 0 \end{cases}$$

We notice that this is exactly the same as the form in the slides. Then, from the slides:

$$\begin{cases} \left[\int_\Omega [\text{grad } N]^T [\text{grad } N]\, d\Omega\right] v - \left[\int_\Omega D^T \hat{N}\, d\Omega\right] p = \left[\int_\Omega N^T \vec{f}\, d\Omega\right] \\ \left[\int_\Omega \hat{N}^T D\, d\Omega\right] v = 0 \end{cases}$$

which can be written as:

$$\boxed{\begin{pmatrix} K & G^T \\ +G & 0 \end{pmatrix} \begin{pmatrix} v \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}}$$