FINITE ELEMENTS IN FLUIDS

ASSIGNMENT4- 2D UNSTEADY TRANSPORT PROBLEM

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HIGH ORDER METHOD

For a high order method a Two-Step Fourth order method has been chosen which is available in the reference book *3.6.4.2* .

$$u(t^{n+1}) = u(t^n) + \Delta t\, u_t^n(t^n) + \frac{\Delta t^2}{2} u_{tt}^n(t^n) + \frac{\Delta t^3}{6} u_{ttt}^n(t^n) + \frac{\Delta t^4}{24} u_{tttt}^n(t^n) + \mathcal{O}(\Delta t^5).$$

By factorization it can be transformed to the following 2-step problem

$$1 + \Delta t \frac{\partial}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3}{\partial t^3} + \frac{\Delta t^4}{24}\frac{\partial t}{\partial t^4} = 1 + \Delta t \frac{\partial}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2}{\partial t^2}\Big(1 + \frac{\Delta t}{3}\frac{\partial}{\partial t} + \frac{\Delta t^2}{12}\frac{\partial^2}{\partial t^2}\Big).$$

This is similar to the 2-Step TG3 method that we are familiar with and yields the following 2-step explicit method-

$$\tilde{u} = u^n + \frac{1}{3}\Delta t\, u_t^n + \frac{1}{12}\Delta t^2\, u_{tt}^n,$$
$$u^{n+1} = u^n + \Delta t\, u_t^n + \frac{1}{2}\Delta t^2\, \tilde{u}_{tt}.$$

The space discritization of these 2 steps are given in the next pages-

First step

$$\tilde{u} = u^n + \frac{1}{3}\Delta t\, u_t^n + \frac{1}{12}\Delta t^2 u_{tt}^n$$

Since diffusion & reaction terms are not considered,

$u_t^n = s - a.\nabla u^n$, &

$u_{tt} = s_t^n - a.\nabla u_t^n = s_t^n - a.\nabla s^n + (a.\nabla)^2 u^n$ [∵ s doesn't depend on time & is only a function of space in our problem]

$\therefore\ \tilde{u}^n = u^n + \frac{1}{3}\Delta t(s - a.\nabla u^n) + \frac{1}{12}\Delta t^2(s_t^n - a.\nabla s^n + a^2\nabla^2 u^n)$

Using galerkin formulation, we have-

$$(\omega, \tilde{u}) = (\omega, u^n) + \frac{1}{3}\Delta t((\omega, s) - a(\omega, \nabla u^n)) + \frac{1}{12}\Delta t^2(-a(\omega, \nabla s^n) + a^2(\omega, \nabla^2 u^n))$$

Integrating by parts,

$$(\omega, \tilde{u}^n) = (\omega, u^n) + \frac{1}{3}\Delta t\left[(\omega, s) - a\{[\omega, u^n]_\Gamma - [\nabla\omega, u^n]\}\right] + \frac{1}{12}\Delta t^2\left[-a\{[\omega, s^n]_\Gamma - [\nabla\omega, s^n]\} + a^2\{[\omega, \nabla u^n]_\Gamma - [\nabla\omega, \nabla u^n]\}\right]$$

$$\Rightarrow \overbrace{(\omega, \tilde{u}^n) = (\omega, u^n)}^{A_1} + \overbrace{\frac{\Delta t}{3}(\omega, s) + \frac{a\Delta t}{3}(\nabla\omega, u^n) + \frac{a\Delta t^2}{12}(\nabla\omega, s^n) - \frac{a^2\Delta t^2}{12}(\nabla\omega, \nabla u^n)}^{B_1} - \left[\frac{a\Delta t}{3}(\omega, u^n)_\Gamma + \frac{a\Delta t^2}{12}(\omega, s^n)_\Gamma - \frac{a^2\Delta t^2}{12}(\omega, \nabla u^n)_\Gamma\right] \rightarrow \Gamma_1$$

Now, $(\omega, \tilde{u}^n) - (\omega, u^n) = \underbrace{(\omega, \Delta\tilde{u}^n)}_{M\Delta\tilde{u}^n} = A_1$

$(\omega, s) = C$

$a(\nabla\omega, u^n) = M_0$

$a(\nabla\omega, s^n) = C_0$

$a^2(\nabla\omega, \nabla u^n) = K$

$a(\omega, u^n)_\Gamma = V_1$

$a(\omega, s^n)_\Gamma = V_2$

$a^2(\omega, \nabla u^n)_\Gamma = V_0$

, $a = (a_x + a_y)$

* B_1 contains all coefficients of u^n.

Second-step

$$u^{n+1} = u^n + \Delta t\, u_t^n + \frac{1}{2}\Delta t^2 \tilde{u}_{tt}^n$$

$$\Rightarrow u^{n+1} = u^n + \Delta t\left(s^n - a.\nabla\tilde{u}^n\right) + \frac{1}{2}\Delta t^2\left(-a.\nabla s^n + (a.\nabla)^2\tilde{u}^n\right)$$

Using galerkin formulation,

$$(\omega, u^{n+1}) = (\omega, u^n) + \Delta t\left((\omega, s^n) - a(\omega, \nabla\tilde{u}^n)\right) + \frac{1}{2}\Delta t^2\left\{-a(\omega, \nabla s^n) + a^2(\omega, \nabla^2\tilde{u}^n)\right\}$$

Integrating by parts,

$$(\omega, u^{n+1}) = (\omega, u^n) + \Delta t\left\{(\omega, s^n) - a\left\{[\omega, \tilde{u}^n]_\Gamma - [\nabla\omega, \tilde{u}^n]\right\}\right\}$$
$$+ \frac{1}{2}\Delta t^2\left\{-a\left\{[\omega, s^n]_\Gamma - [\nabla\omega, s^n]\right\} + a^2\left\{[\nabla\omega, \nabla\tilde{u}^n]_\Gamma - [\nabla\omega, \nabla\tilde{u}^n]\right\}\right\}$$

$$\Rightarrow \overbrace{(\omega, u^{n+1})}^{A_2} = (\omega, u^n) + \Delta t(\omega, s^n) + a\Delta t(\nabla\omega, \tilde{u}^n) + \frac{a\Delta t^2}{2}(\nabla\omega, s^n) - \frac{a^2\Delta t^2}{2}(\nabla\omega, \nabla\tilde{u}^n)$$
$$- \left[a\Delta t(\omega, \tilde{u}^n)_\Gamma + \frac{a\Delta t^2}{2}(\omega, s^n)_\Gamma - \frac{a^2\Delta t^2}{2}(\omega, \nabla\tilde{u}^n)_\Gamma\right]$$

* All terms ~~with~~ having coefficients of $\tilde{u}^n$ is under C2.

$B_2 = \Delta t \cdot C - \Delta t M_0$

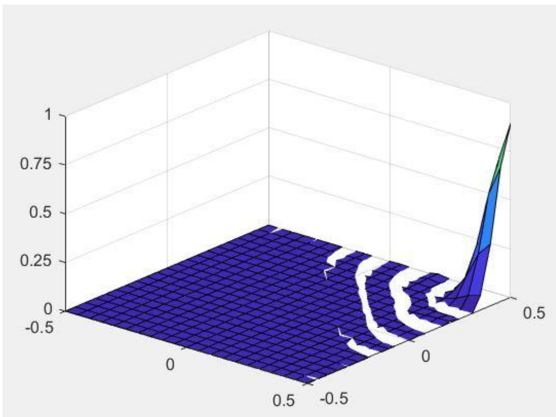
f_2 = rest

Since the code is similar to 2-step TG3, with essentially different value of alpha, the same variables can be used for this method also.

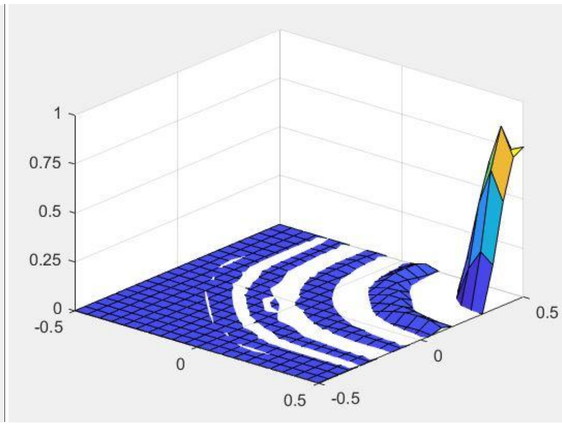
```
elseif meth==8
    alpha = 1/12;
    A1 = M;
    B1 = -(dt/3)*C'- alpha*dt^2*(K - Co);
    f1 = (dt/3)*v1 + alpha*dt^2*(v2 - vo);
    A2 = M;
    B2 = -dt*C';
    C2 = - (dt^2/2)*(K-Co);
    f2 = dt*v1 - (dt^2/2)*(v2 - vo);
```

Behavior of methods

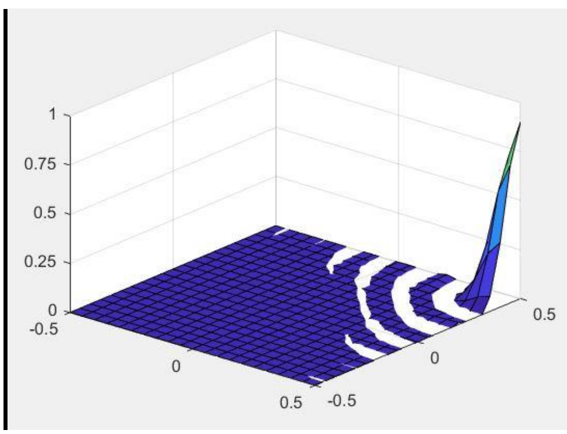
The results for different methods will be obtained for third problem because of its complexity and better profile than the rest, for 20 elements in x and y axis and the default values of other variables.



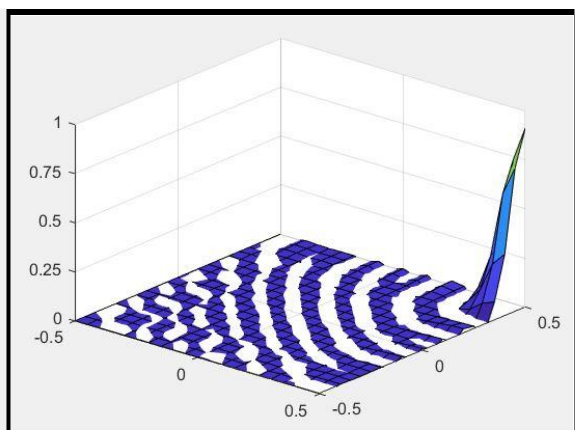
TG-2



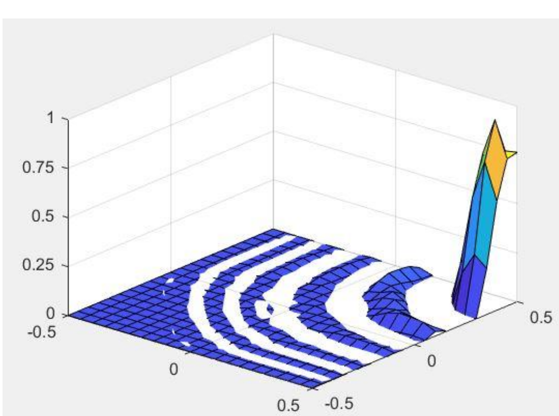
LW-FD



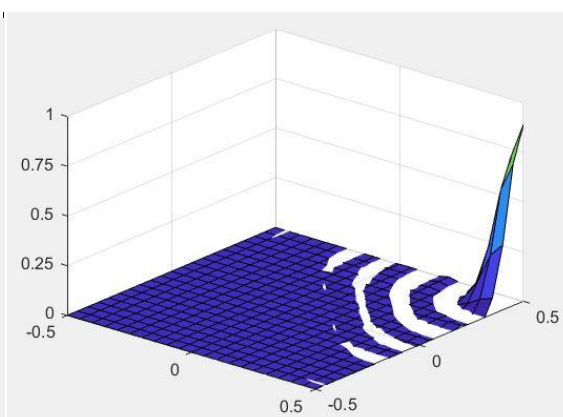
TG-3



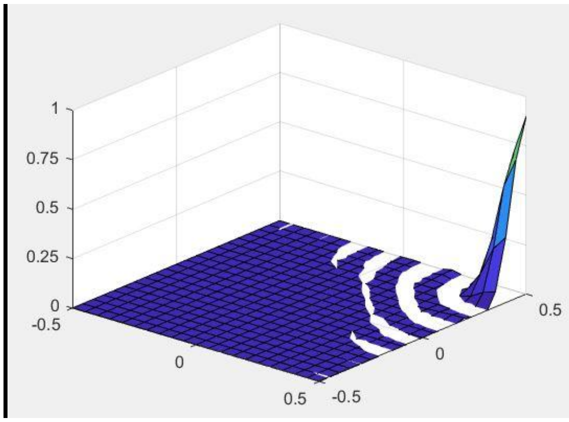
CN



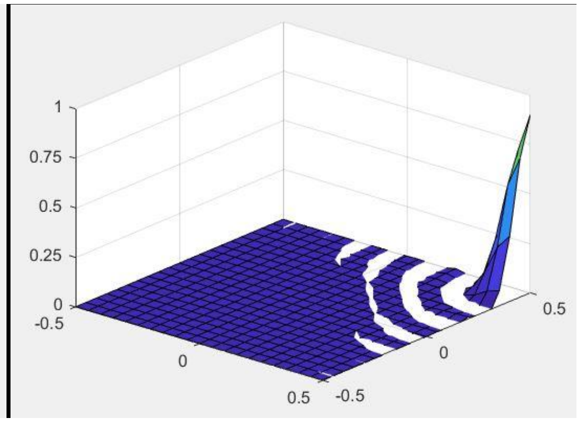
CN-FD



CJ



TG3-2S



2S-4O

We can see from the above plots that the higher order methods TG3, TG3-2S and 2S-4O give almost the same results and although it is hard to notice, the oscillations are a little smoother than TG-2 because TG-2 has lower order of accuracy than the above methods. This difference is clearer in the other problems.

We also notice that the other implicit methods produce more oscillations compared to the higher order methods which is to be expected given the lower order of accuracy. Crank Nicholson +Galerkin (CN) produces more oscillations than the other methods. This is consistent with the theory provided the non-dissipative nature in pure convection and decreasing accuracy with increasing time step.