1D and 2D Unsteady Transport
Finite element in fluids

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1 1D convective transport

1.1 Stability analysis

In this section, the stability of four different methods is studied for the 1D convection equation without source term and simple initial conditions. The methods studied for time discretization are: Lax-Wendroff (LW) with and without lumped mass matrix (LW-FD), Crank-Nicolson (CN) and Third order Taylor-Galerkin. The spatial discretization is done using Galerkin method. The results obtained using 120 time steps and convection coefficient $a = 1$ for three different initial conditions are depicted in the figures below. The figures show the expected results: LW is stable for $C^2 < \frac{1}{3}$, LW-FD for $C^2 < 1$, CN is unconditionally stable and TG3 for $C^2 < 1$. Moreover, we can see that the accuracy of the solution depends on C and the dimensionless wave number. The third order Taylor-Galerkin is the method that shows better accuracy of the four studied.

1. Initial condition 1:

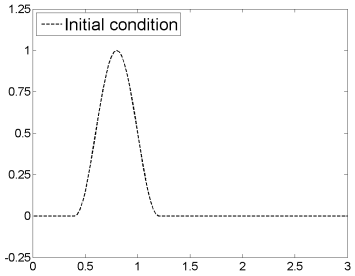
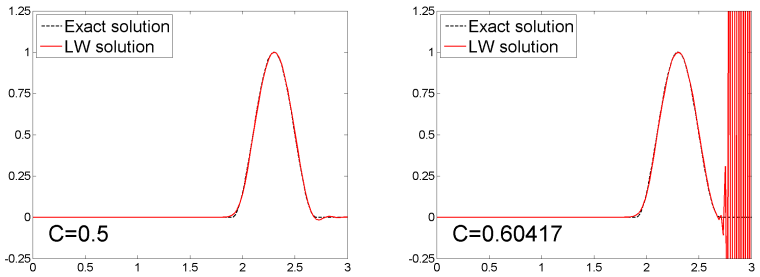
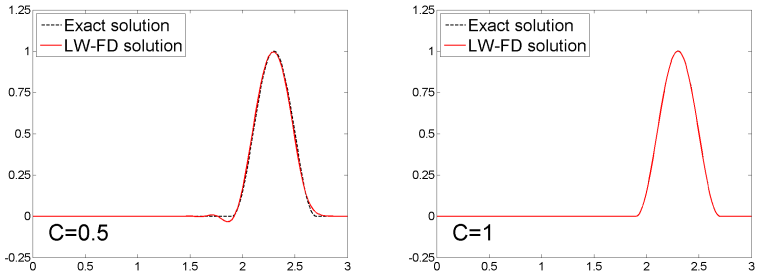


Figure 1: Initial conditions 1

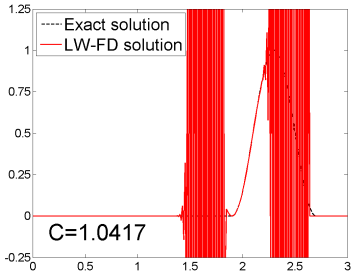


(a) (b)

Figure 2: Lax-Wendrof for C=0.5 and C=0.6

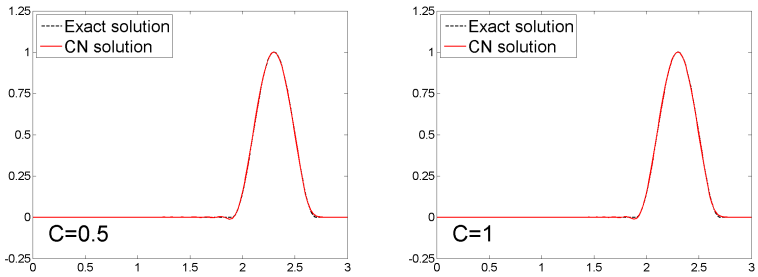


(a) (b)



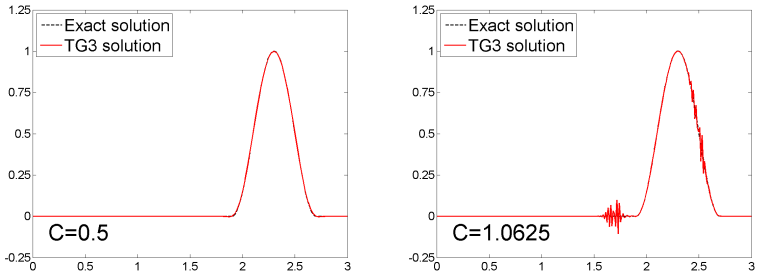
(c)

Figure 3: Lax-Wendrof with lumped matrix for C=0.5 and C=1 and $C > 1$



(a) (b)

Figure 4: Crank-Nicholson for C=0.5 and C=1



(a) (b)

Figure 5: Third order Taylor-Galerkin for C=0.5 and $C > 1$

2. Initial condition 3:

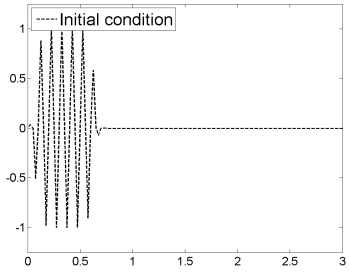
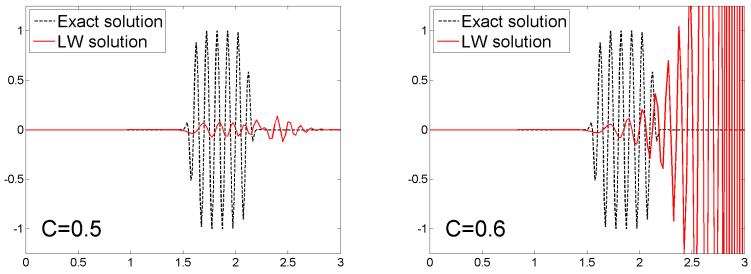
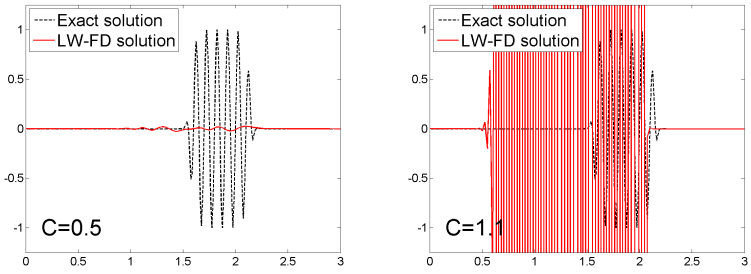


Figure 6: Initial conditions 3



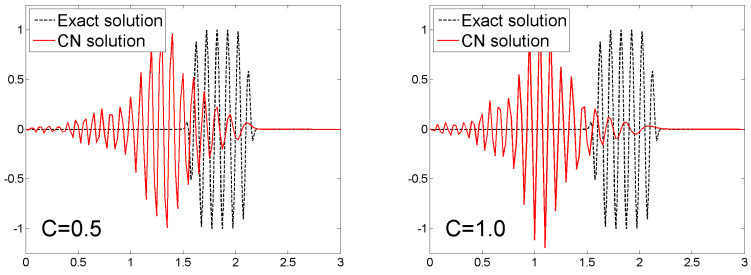
(a) (b)

Figure 7: Lax-Wendrof for C=0.5 and C=0.6 and initial conditions 3



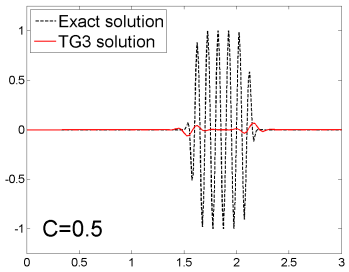
(a) (b)

Figure 8: Lax-Wendrof with lumped matrix for C=0.5 and $C > 1$ and initial conditions 3

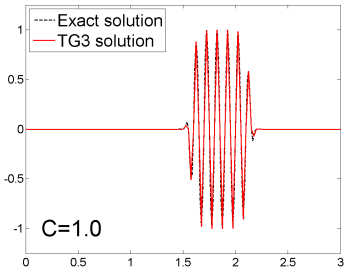


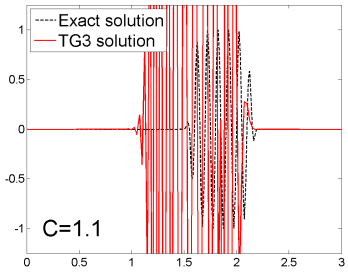
(a) (b)

Figure 9: Crank-Nicholson for C=0.5 and C=0.9 and initial conditions 3



(a)





(b) (c)

Figure 10: Third order Taylor-Galerkin for C=0.5, C=0.9 and $C > 1$ and initial conditions 3

3. Initial condition 4:

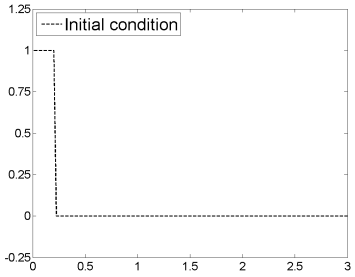
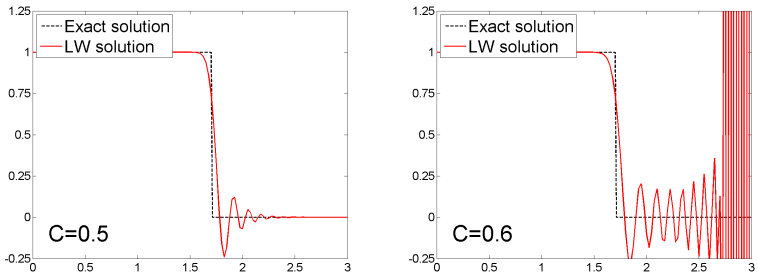
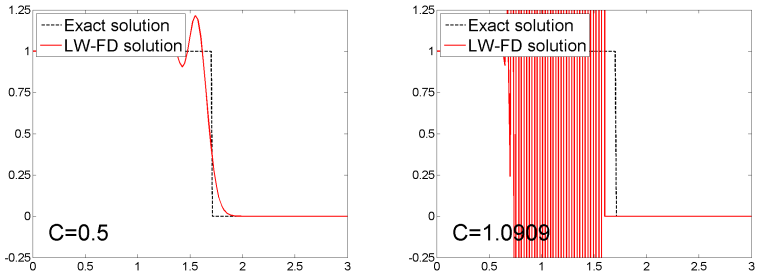


Figure 11: Initial conditions 4



(a) (b)

Figure 12: Lax-Wendrof for C=0.5 and C=0.6 and initial conditions 4



(a) (b)

Figure 13: Lax-Wendrof with lumped matrix for C=0.5 and $C > 1$ and initial conditions 4

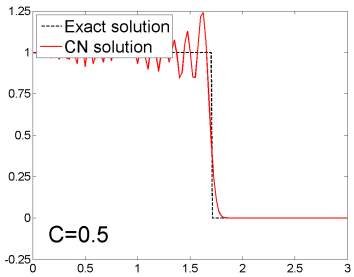
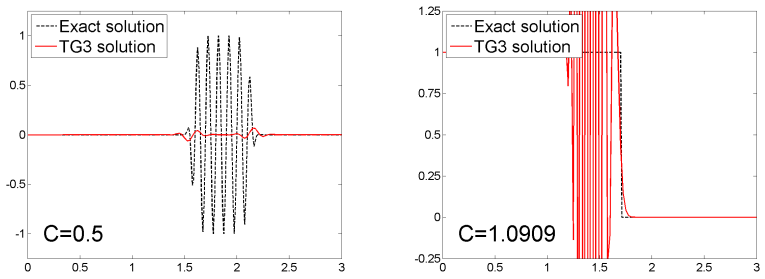


Figure 14: Crank-Nicholson for C=0.5 and initial conditions 4



(a) (b)

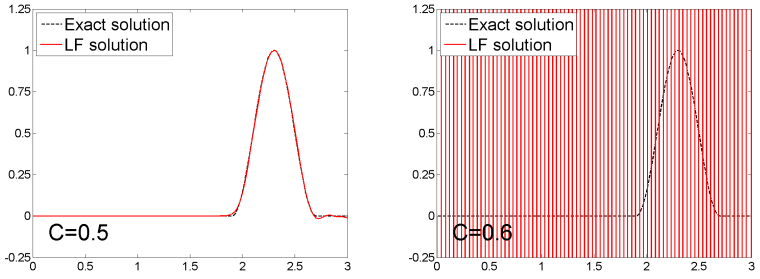
Figure 15: Third order Taylor-Galerkin for C=0.5 and $C > 1$ and initial conditions 4

1.2 Leap-frog method

The leap-frog method has also been implement for a 0 source term:

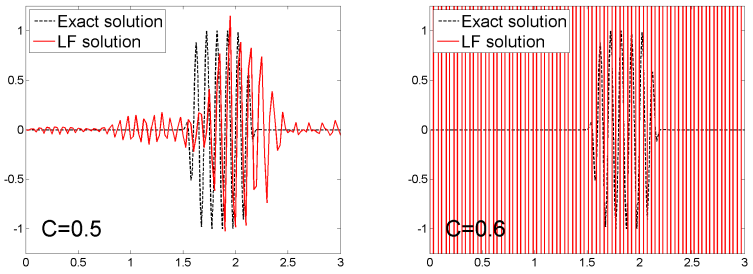
$$u^{n+1} = u^{n-1} - 2\Delta t \boldsymbol{a} \cdot \boldsymbol{\nabla} \boldsymbol{u^n}$$

The results obtained for different values of C and the initial conditions studied in section 1.1 are depicted in the following figures. As can be seen in the figures, the method is stable for C=0.5 but not for C=0.6 (its theoretical stability limit is C=0.57). Moreover, this method shows better accuracy than the methods seen in section 1.1 (with exception of the third order Taylor-Galerkin method), especially for initial conditions 3.



(a) (b)

Figure 16: Leap-frog method for C=0.5 and C=0.6 and initial conditions 1



(a) (b)

Figure 17: Leap-frog method for C=0.5 and C=0.6 and initial conditions 3

2 1D unsteady convection-diffusion equationt

In this section, a diffusion term has been added to the equation. The initial solution has been imposed to be:

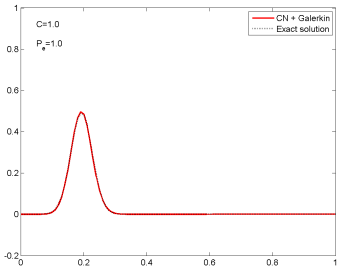
$$u\left(x,0\right)=\frac{5}{7}\exp\left(-\left(\frac{x-x_0}{L}\right)^2\right)$$

The analytical solution for this case is:

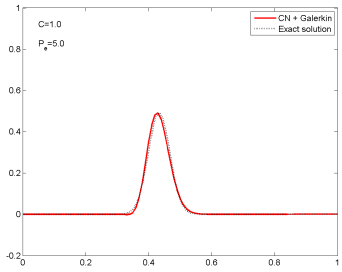
$$u\left(x,t\right)=\frac{5}{7\sigma}\exp\left(-\left(\frac{x-x_0-at}{\sigma L}\right)^2\right)$$

$$\sigma=\sqrt{1+\frac{4\nu t}{L^2}}\quad x_0=\frac{2}{15}\quad L=\frac{7\sqrt{2}}{300}$$

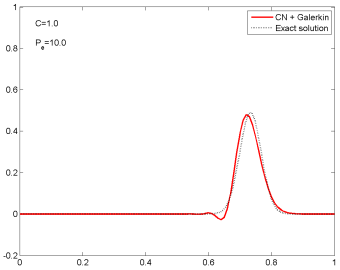
The influence of both Pe and C has been used for Crank-Nicholson method for time and a Galerkin formulation for space. The values used by default to obtain Pe = 1 and C = 1 are: $a=0.2$, $\nu=0.001$, nº of elements = 100, time step = 0.05 and end time = 0.3. In order to work with different Pe and C, the values of the time step and a are changed. The effect of increasing Peclet number is depicted in Fig. 18 and for Courant number is depicted in Fig. 19. As is depicted in the figures, increasing the Peclet number we have a faster transport, but no instabilities are introduced. However, when increasing Courant's number, small instabilities appear.



(a)

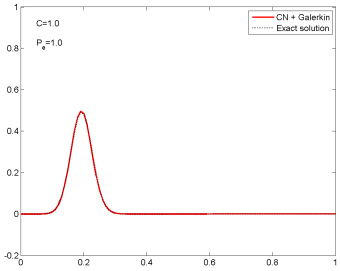


(b)

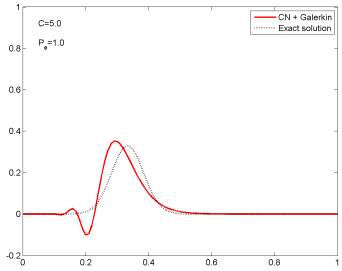


(c)

Figure 18: Effects of Peclet number



(a)



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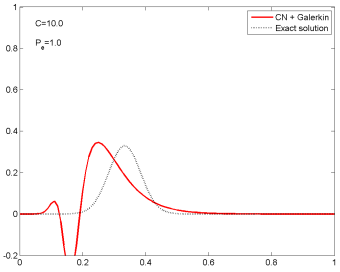


(c)

Figure 19: Effects of Courant number

3 2D convection equation

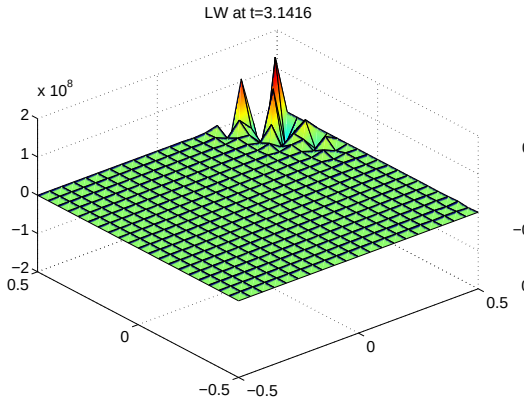
A code has been implemented to solve for the 2D convection equation using a Galerkin spatial discretization and Lax-Wendroff, Lax-Wendroff with lumped mass matrix and Crank-Nicholson methods for the temporal discretization. The convection field used is a pure rotation one and the initial conditions of the problem are:

$$u = (\boldsymbol{x}, 0) = \begin{cases} \frac{1}{4}\left(1 + cos\pi X_1\right)\left(1 + cos\pi X_2\right) & if X_1^2 + X_2^2 \leq 1, \\ 0 & otherwise \end{cases}$$

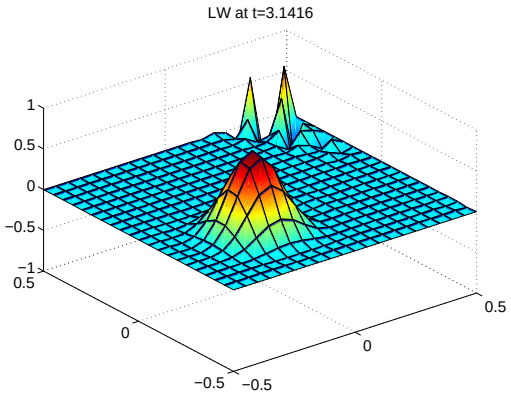
The results obtained using a mesh of 20x20 bilinear quadrilateral elements and a final time $t_{end} = 2\pi$ are depicted in the following figures. In Fig. 20 it can be apreciated that the Lax-Wendroff method show unstabilites when using 120 and 150 time steps. When using a lumped mass matrix, the method does not present unstabilites for these number of time steps, but the results obtained are very poor (see Fig. 21b). In the case of the Crank-Nicholson method, no instabilites appear when using 120 time steps and a good accuracy is obtained. The maximum values of u obtained at $t = 2\pi$ are:

$$\begin{cases} \text{For LW and 120 time steps:} & u_{max} = 0.911601 \\ \text{For LWFD and 120 time steps:} & u_{max} = 0.624378 \\ \text{For CN and 120 time steps:} & u_{max} = 0.995201 \end{cases}$$

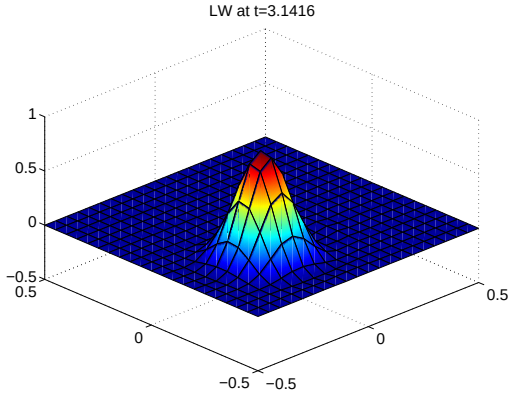
It can be concluded that the explicit methods used (Lax-Wendroff with and without lumped mass matrix) are disipative, while the Crank-Nicholson method does not show dissipation and obtains a value of u very close to 1. The results obtained could be improved refining the mesh.



(a) 120 time steps

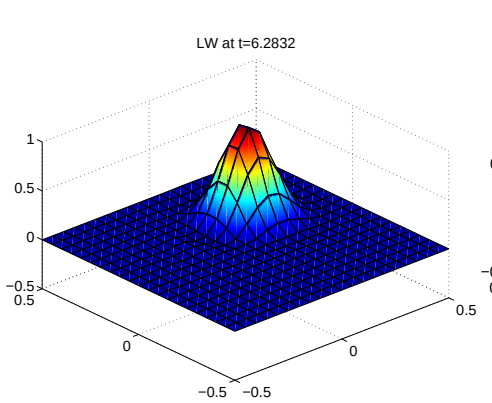


(b) 150 time steps

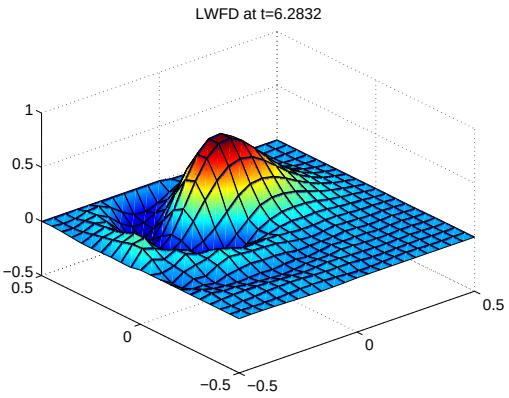


(c) 200 time steps

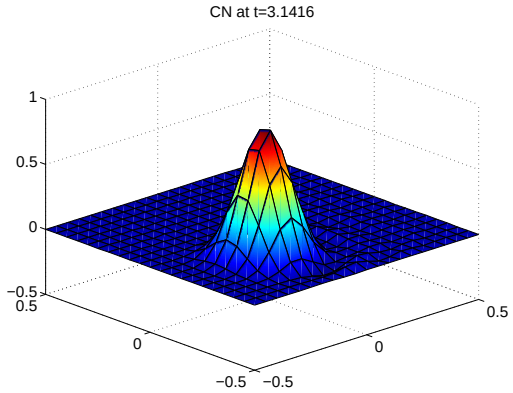
Figure 20: Results at $t = \pi$ using Lax-Wendroff method with a mesh of 20x20 bilinear quadrilateral elements



(a) Lax-Wendroff using 200 time steps



(b) Lax-Wendroff with lumped mass matrix using 200 time steps



(c) Crank-Nicholson method using 120 time steps

Figure 21: Comparison of the results at $t = 2\pi$ for a mesh of 20x20 bilinear quadrilateral elements