1D and 2D Steady transport equations Finite element in fluids

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Abstract

In this report, the results obtained for five different methods (Galerkin, SU, SUPG, GLS and SGS) are compared to solve for the 1D steady convection-diffusion equation with different Peclet number, stabilization parameter, linear and quadratic elements and mesh refinement. After that, the code has been improved to account for the effects of the reaction term and the effects of this term are studied. Then the 2D convection-diffusion equation is solved with Neumann and Dirichlet boundary conditions and the performance of the different methods tested is compared. Finally, a reaction term is added to the 2D problem and a convection-reaction dominated case and a reaction dominated problem are studied.

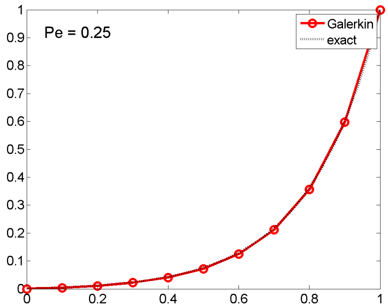
1 1D convective diffusion equation

1.1 Linear elements

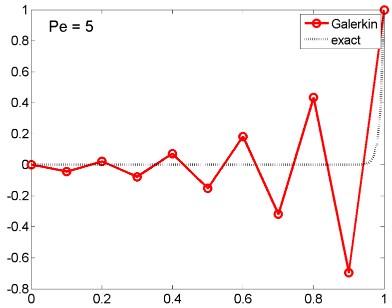
First, I used Galerkin's method for different values of a, υ, nº of elements and s=0 (Fig. 1) The results obtained show that the Galerkin method provides good results for diffusion dominated problems ($Pe \leq 1$). However, for $Pe > 1$, the method is unstable and the results obtained are not acceptable.

In order to improve the results for convection-dominated problems, stabilization methods are needed. Below you can find the results obtained using SU, SUPG, GLS and SGS methods for a=1, υ=0.01 and 10 linear elements and using the optimal stabilization parameter (Fig. 2). For these methods, the exact solution is obtained at the nodes. However, with the number of elements used, the evolution of the solution at the boundary layer cannot be captured.

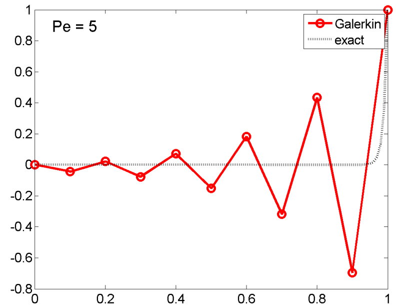
The effect of the stabilization parameter for the SUPG method is studied next. As can be seen, the stabilization parameter can change the solution completely, ranging from a solution with too much artificial diffusion (Fig. 4a) to the solution obtained for Galerkin's method (Fig. 4c).



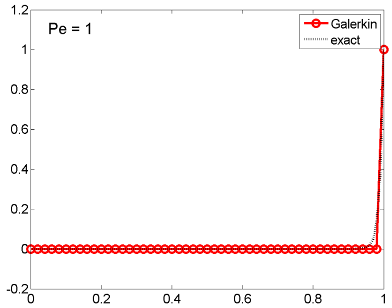
(a) a=1,υ=0.2 and 10 elements



(b) a=20,υ=0.2 and 10 elements

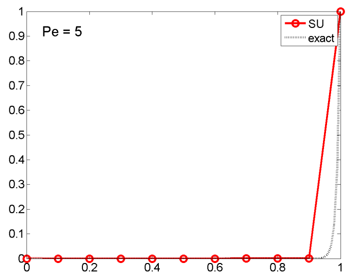


(c) a=1,υ=0.01 and 10 elements

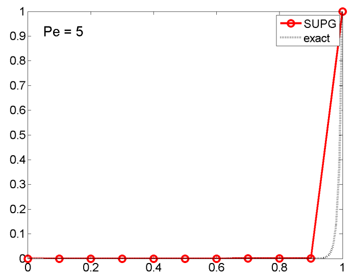


(d) a=1,υ=0.01 and 50 elements

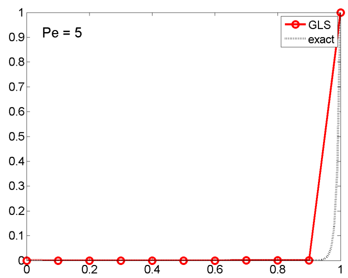
Figure 1: Results for Galerkin's method with linear elements



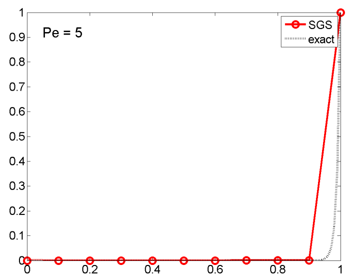
(a) SU method



(b) SUPG method

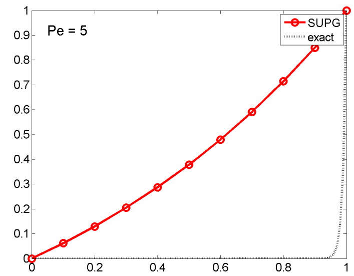


(c) GLS method

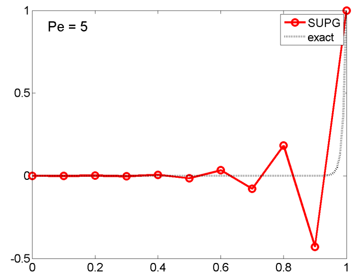


(d) SGS method

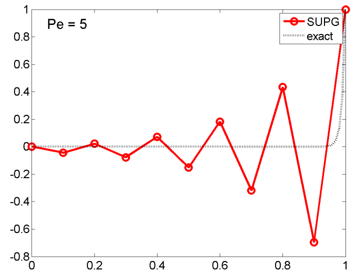
Figure 2: Results for a=1,υ=0.01 and 10 linear elements using different stabilization methods



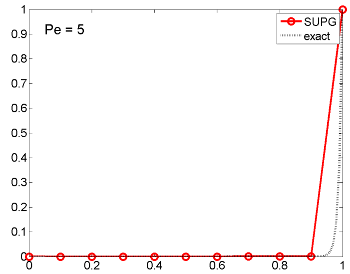
(a) τ=1



(b) τ=0.01



(c) τ=0



(d) Optimal value of τ

Figure 3: Result for a=1,υ=0.01 and 10 linear elements and the SUPG method with different values of τ

1.2 Quadratic elements

In this section quadratic elements are studied. To use the stabilization methods seen above, two different τ must be defined:

At the middle node:

$$\beta_m = cothPe - \frac{1}{Pe}$$

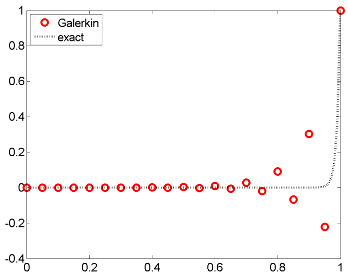
$$\tau_m = \beta_m \frac{h}{2a}$$

At the corner nodes:

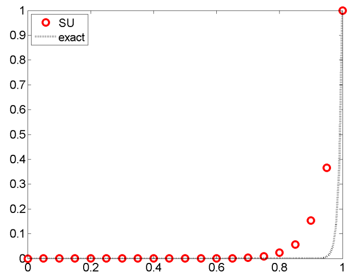
$$\beta_c = \frac{(2Pe-1) + (-6Pe+7)\, e^{-2Pe} + (-6Pe-7)\, e^{-4Pe} + (2Pe+1)\, e^{-6Pe}}{(Pe+3) + (-7Pe-3)\, e^{-2Pe} + (7Pe-3)\, e^{-4Pe} - (Pe+3)\, e^{-6Pe}}$$

$$\tau_c = \beta_c \frac{h}{2a}$$

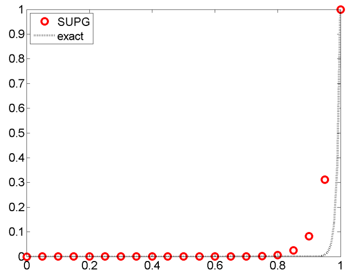
The problem is solved again for a=1, υ=0.01 and 10 linear elements using the 5 different methods studied above (see Fig. 4). The results obtained show that the oscillations obtained with Galerkin's method are smaller and the shape of the result obtained with the 4 stabilization methods is more similar to the shape of the exact solution. However, the results obtained at the nodes are not exact and nodal results obtained in the region with a higher gradient are worst than those obtained with linear elements.



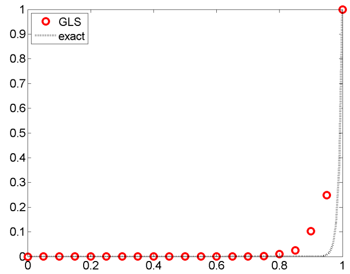
(a) Galerkin



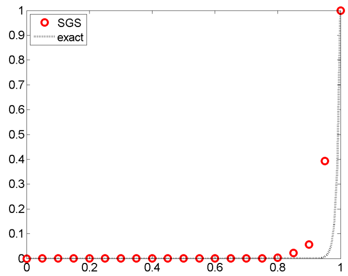
(b) SU



(c) SUPG



(d) GLS



(e) SGS

Figure 4: Result for a=1,υ=0.01 and 10 quadratic elements

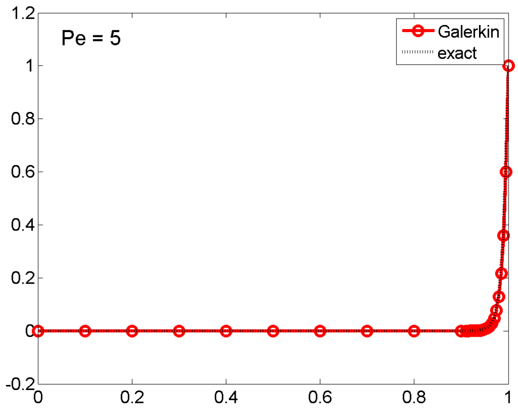


Figure 5: Result using Galerkin's method for a=1,υ=0.5, σ=0 and refined sharp front

1.3 Galerkin's method improvement: Refinement of the sharp front

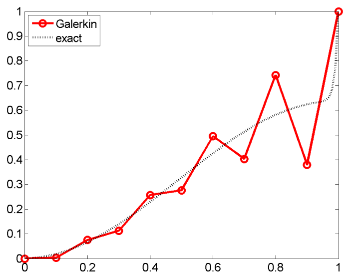
Galerkin method's results can be improved if the sharp front is refined. The results obtained for a=1, υ=0.01, linear elements and using an element size of 0.01 in the sharp front are depicted in Fig. 5. The improvements are obvious if compared with the results obtained without refining the mesh in the sharp front (see Fig. 1).

1.4 Source term

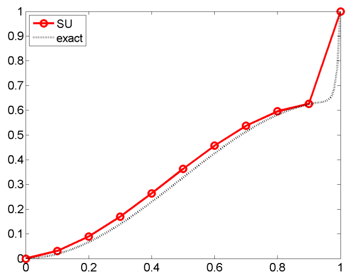
In this section, the problem is solved adding the following source term:

$$s = \sin(\pi x)$$

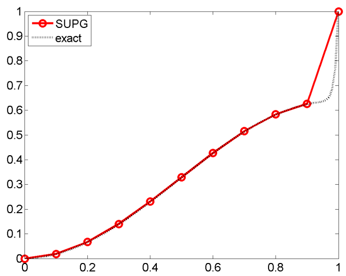
The results obtained are depicted in Fig. 6. Since $Pe = 5$, Galerkin's method provides unstables results. The rest of the methods provide stable results, but the SU methods does not provide the exact solution at the nodes.



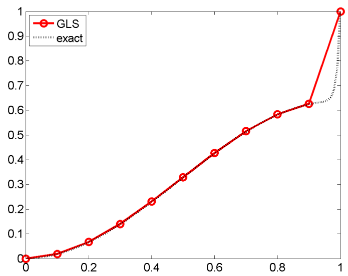
(a) Galerkin



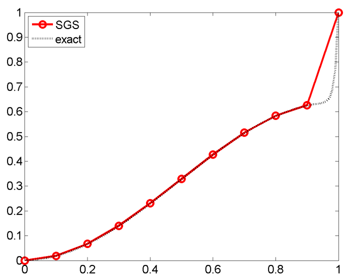
(b) SU



(c) SUPG



(d) GLS



(e) SGS

Figure 6: Result for a=1,υ=0.01 and 10 linear elements

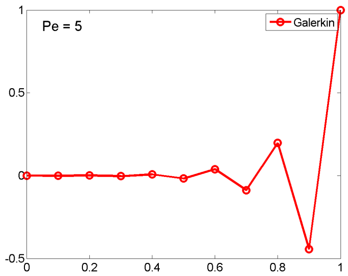
2 1D steady convection-diffusion-reaction case

In this section, a reaction term is added to the problem. In the weak form, this can be done adding the term $\int_\Omega N^T \sigma N d\Omega$. The stabilization coefficient used is:

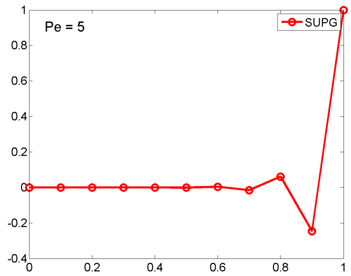
$$\tau = \frac{h}{2a}\left(1 + \frac{9}{Pe^2} + \left(\frac{h\sigma}{2a}\right)^2\right)^{-1/2}$$

The results obtained with the different methods for a=1,υ=0.01, σ=20 and 10 linear elements are depicted in Fig 7. The results obtained with the Galerkin are totally oscillatory. SUPG and GLS methods show small oscillations near the boundary layer. SGS is affected by the presence of the reaction term and does not oscillate.

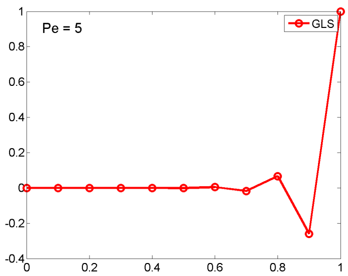
In the last examples the effects of the reaction could not be seen due to the high value of the Peclect number. If using Galerkin's method for a=1,υ=0.5, σ=-20 and 10 linear elements, the results obtained show the effects of the reaction (see Fig. 8). The first one has reaction, the second one does not have it).



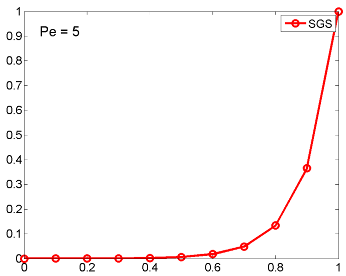
(a) Galerkin



(b) SUPG



(c) GLS



(d) SGS

Figure 7: Result for a=1,υ=0.01, σ=20 and 10 linear elements

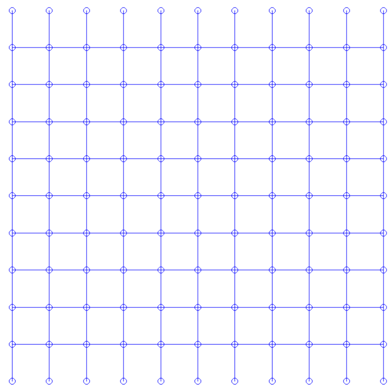
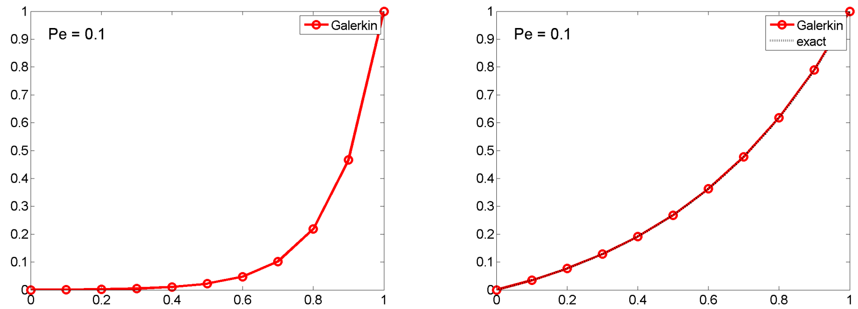


Figure 9: Mesh for the 2D case



(a) a=1,υ=0.5, σ=20 (b) a=1,υ=0.5, σ=0

Figure 8: Effects of the reaction term

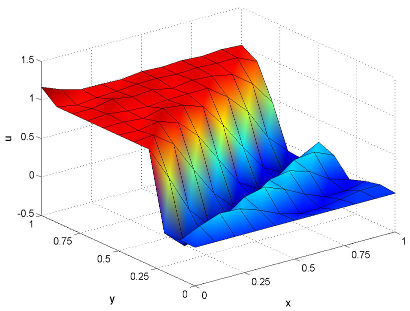
3 2D steady convection-diffusion equation

In this section, the 2D steady convection-diffusion equation is solved in a 1x1 square domain using the mesh depicted in Fig. 9.

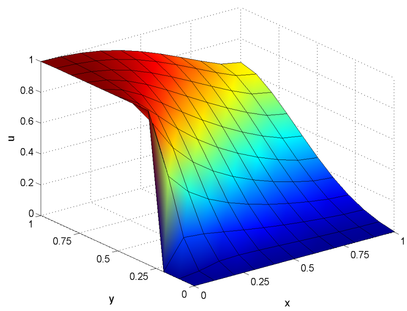
First, the problem has been solved with Dirichlet boundary conditions in half of the boundary, imposing 1 at the boundary $(x = 0, y \in (0.2, 1])$ and 0 at $(x, y = 0)$ and $(x = 0, y \in [0, 2])$. Homogeneous natural boundary conditions are imposed at the rest of the boundary (outlet). The results obtained for $\|a\| = 1$ and $\upsilon = 10^{-4}$ (Pe = 500) using Galerkin's, Aritificial Diffusion and SUPG methods are depicted in Fig 10. As is showed in the figures, Galerkin method cannot satisfactorily resolve the discontinuity and produce spurious oscillations and Artificial diffusion method introduces too much crosswind diffusion. The results obtained with SUPG method are better.

After that, the problem was solved changing the Neumann boundary conditions imposed at $(x = 1, y = 1)$ by homogeneous Dirichlet boundary conditions. The results obtained are depicted in Fig. 11. As is depicted in the figures, the result obtained with Galerkin's method is completely unstable and does not represent the real solution. The result obtained adding artificial diffusion is better and smoother, but too much diffusion is added. The result obtained with SUPG method is better than Galerkin's and does not add too much diffusion, however, the mesh used shows problems to capture all the details of the solution, especially at the boundaries.

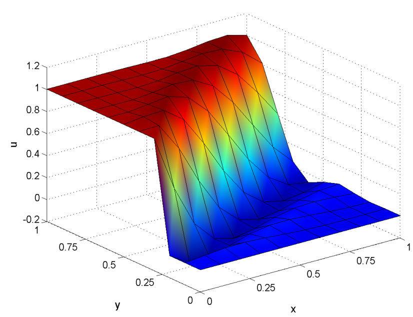
Finally, a reaction term was added to the code.To add the reaction term, I added $\int_\Omega N^T \sigma N d\Omega$ to the global matrix. For SUPG, an extra term multiplying τ was needed. Two different cases were solved with zero Dirichlet boundary conditions at the entire boundary and a constant source term s=1: a convection-reaction dominated problem with $\|a\| = \frac{1}{2}$, $\upsilon = 10^{-4}$ and $\sigma = 1$ (Pe = 250) and a reaction dominated case with $\|a\| = 10^{-3}$, $\upsilon = 10^{-4}$ and $\sigma = 1$ (Pe = 0.5). The results obtained for the convection-reaction dominated case are depicted in Fig. 12. In this case, srong oscillations appear for Galerkin's method due to the high Peclet number, but the results obtained using both artificial diffusion and SUPG method are acceptable, specially taking into account that a coarse mesh is being used. The results obtained with the three different methods used for the reaction dominated case (Fig. 13) are very similar because we are working with $Pe < 1$. The results at the boundary should be improved, either using a finer mesh or a different method.



(a) Galerkin

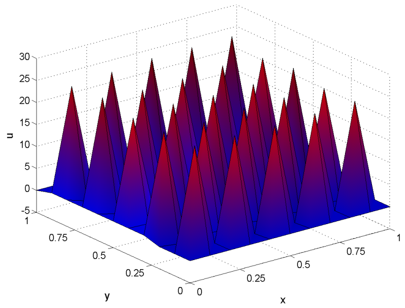


(b) Artificial diffusion

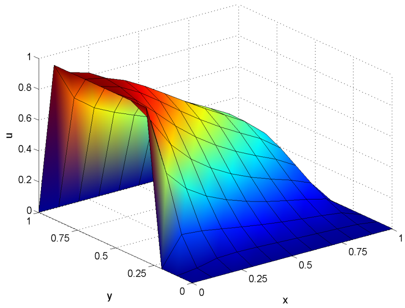


(c) SUPG

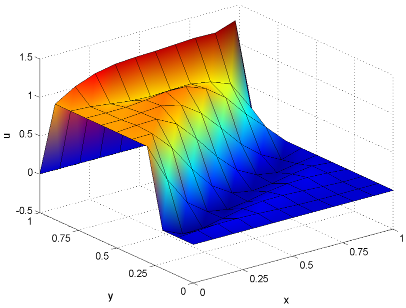
Figure 10: Results for $\upsilon = 10^{-4}$, $\|a\| = 1$, Dirichlet bc of 0 and 1 and homogeneous Neumann bc at the outlet



(a) Galerkin

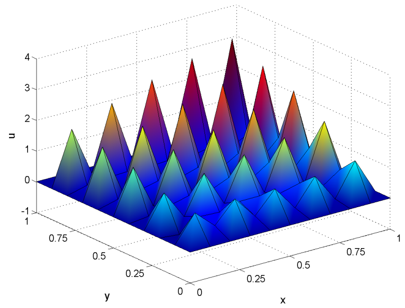


(b) Artificial diffusion

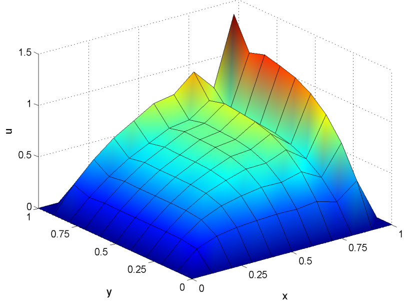


(c) SUPG

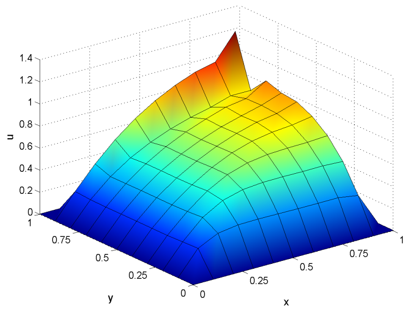
Figure 11: Results for $\upsilon = 10^{-4}$, $\|a\| = 1$, Dirichlet bc of 0 and 1 at the inlet and 0 at the outlet



(a) Galerkin

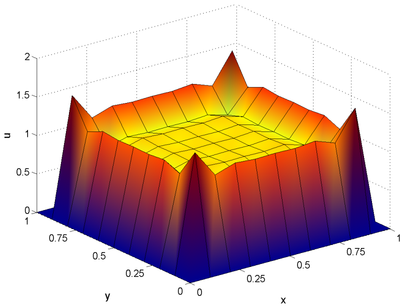


(b) Artificial diffusion

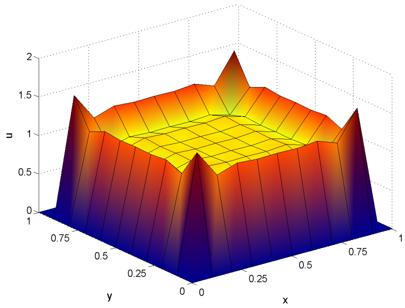


(c) SUPG

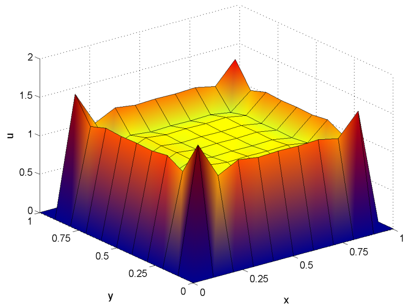
Figure 12: Results for the convection-reaction dominated problem



(a) Galerkin



(b) Artificial diffusion



(c) SUPG

Figure 13: Results for the reaction dominated problem