FINITE ELEMENT METHODS IN FLUIDS

Master of Science in Computational Mechanics 2015-2016

ASSIGNMENT

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Exercise 1

Part A.

$$\mathbf{a} \cdot \boldsymbol{\nabla}\rho - \boldsymbol{\nabla} \cdot (\nu\boldsymbol{\nabla}\rho) + \sigma \cdot \rho = \mathrm{s} \quad \text{in } \Omega$$

Multiplying by a test function ω yields:

$$\int_\Omega \omega(\mathbf{a} \cdot \boldsymbol{\nabla}\rho)\mathrm{d}\Omega - \int_\Omega \omega\boldsymbol{\nabla} \cdot (\nu\boldsymbol{\nabla}\rho)\mathrm{d}\Omega + \int_\Omega \omega\sigma \cdot \rho\, \mathrm{d}\Omega = \int_\Omega \omega \mathrm{s}\, \mathrm{d}\Omega$$

Integrating by parts, the weak form of a problem is obtained:

$$\int_\Omega \omega(\mathbf{a} \cdot \boldsymbol{\nabla}\rho)\mathrm{d}\Omega + \int_\Omega \boldsymbol{\nabla}\omega \cdot (\nu\boldsymbol{\nabla}\rho)\mathrm{d}\Omega + \int_\Omega \omega\sigma \cdot \rho\, \mathrm{d}\Omega = \int_\Omega \omega \mathrm{s}\, \mathrm{d}\Omega + \int_{\Gamma_\mathrm{N}} \omega\rho_\mathrm{N}\, \mathrm{d}\Gamma$$

for all ω

Discretization of the weak form using Galerkin approximation $\rho^h(x) = \sum \rho_j\, N_j(x)$ yields:

$$\int_\Omega \omega^h(\mathbf{a} \cdot \boldsymbol{\nabla}\rho^h)\mathrm{d}\Omega + \int_\Omega \boldsymbol{\nabla}\omega^h \cdot (\nu\boldsymbol{\nabla}\rho^h)\mathrm{d}\Omega + \int_\Omega \omega^h\sigma \cdot \rho^h\, \mathrm{d}\Omega = \int_\Omega \omega^h \mathrm{s}\, \mathrm{d}\Omega + \int_{\Gamma_\mathrm{N}} \omega^h{\rho^h}_\mathrm{N}\, \mathrm{d}\Gamma$$

Part B. The problem was solved using quadrilateral elements with $h = 0.2$. The convective term a is in the x direction.

The MATLAB code was modified in order to include the reaction term and impose Dirichlet boundary conditions $\rho = 0$ in Γ_4, $\rho = 1$ in Γ_2 and Neumann boundary conditions in all other parts of the boundary.

Case 1. The parameters of the this case are the following: $||\mathrm{a}|| = 1, \nu = 0.001,$ 10 linear elements in each direction, $\sigma = 10^{-3}, \mathrm{s} = 0.$ The solution for ρ has spurious oscillations, see Figure 1.

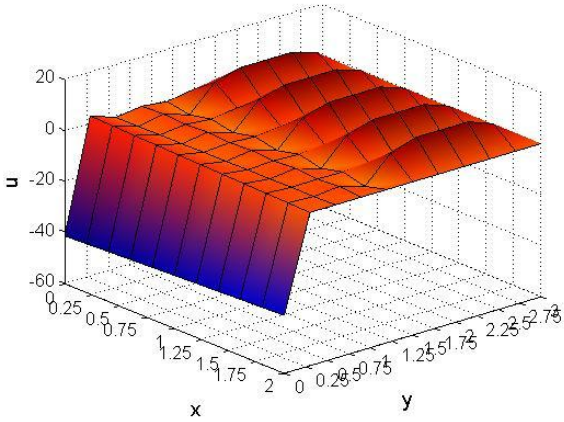

Figure 1. The solution of case 1 using Galerkin space discretication

The reason for this oscillations is a high Péclet number.

Case 2. The parameters of the this case are the following: $||a|| = 10^{-3}, \nu = 0.001$, 10 linear elements in each direction, $\sigma = 1, s = 0$.

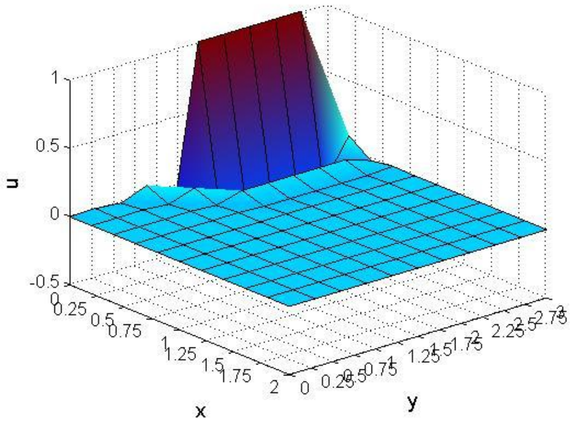

Figure 2. The solution of case 2 using Galerkin space discretication

Case 3. The parameters of the this case are the following: $||a|| = 1, \nu = 0.001$, 10 linear elements in each direction, $\sigma = 0, s = 1$. The solution for ρ has spurious oscillations, see Figure 1.

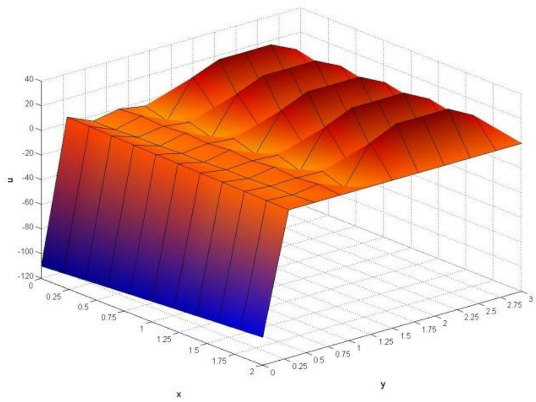

Figure 3. The solution of case 3 using Galerkin space discretication

For the same reason as in the case 1, spurious oscillations appear, which are amplified by a source term.

Case 4. The parameters of the this case are the following: $||a|| = 1, \nu = 0.001$, 10 linear elements in each direction, $\sigma = 1, s = 0$

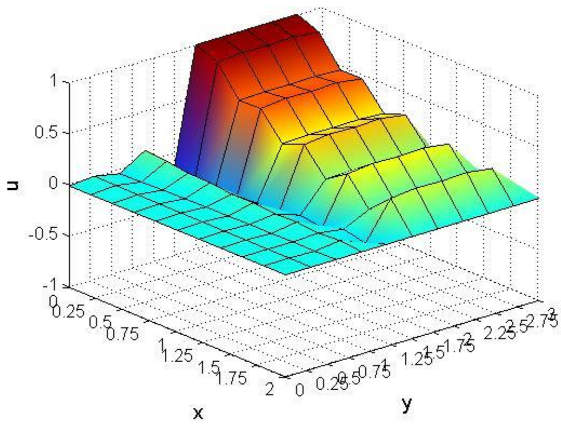

Figure 4. The solution of case 4 using Galerkin space discretication

Part C. In Case 2 the Péclet number is small $Pe = 1$, comparing to $Pe = 50$, in other cases. Therefore stabilization effect would not be so clearly seen in Case 2. Moreover, it is reaction-dominated. For the reaction-dominated cases Galerkin, SUPG, GLS give similar results **[1, p. 77]**. So for cases 2 and 4 SGS is the most effective method of solution. As we are interested in comparing Galerkin, SUPG and GLS solutions, the case 1 was chosen for the studying the influence of this methods. However, Case 3 also has sense to be used for these purposes.

Derivation and implementation of SUPG. We start from a weak form of the problem:

$$\int_{\Omega} \omega^h(\mathbf{a} \cdot \boldsymbol{\nabla}\rho^h)d\Omega + \int_{\Omega} \boldsymbol{\nabla}\omega^h \cdot (\nu\boldsymbol{\nabla}\rho^h)d\Omega + \int_{\Omega} \omega^h\sigma \cdot \rho^h \, d\Omega = \int_{\Omega} \omega^h s \, d\Omega \int_{\Gamma_N} \omega^h {\rho^h}_N \, d\Gamma$$

We introduce consistent stabilization term:

$$\sum_{e} \int_{\Omega} (\boldsymbol{a} \cdot \boldsymbol{\nabla}\rho^h)\tau((\boldsymbol{a} \cdot \boldsymbol{\nabla}\rho^h) - \boldsymbol{\nabla} \cdot (\nu\boldsymbol{\nabla}\rho^h) + \sigma \cdot \rho^h - s)d\Omega$$

Unlike Streamline Upwind (SU), we multiplied all the terms by stabilization parameter and therefore developed consistent stabilization technique:

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$$\int_{\Omega} \omega^h(\mathbf{a} \cdot \boldsymbol{\nabla}\rho^h)d\Omega + \int_{\Omega} \boldsymbol{\nabla}\omega^h \cdot (\nu\boldsymbol{\nabla}\rho^h)d\Omega + \int_{\Omega} \omega^h\sigma \cdot \rho^h \, d\Omega +$$

$$+\sum_{e} \int_{\Omega} (\boldsymbol{a} \cdot \boldsymbol{\nabla}\rho^h)\tau((\boldsymbol{a} \cdot \boldsymbol{\nabla}\rho^h) - \boldsymbol{\nabla} \cdot (\nu\boldsymbol{\nabla}\rho^h) + \sigma\rho^h)d\Omega = \int_{\Omega} \omega^h s \, d\Omega + \sum_{e} \int_{\Omega} (\boldsymbol{a} \cdot \boldsymbol{\nabla}\rho^h)\tau s d\Omega + \int_{\Gamma_N} \omega^h {\rho^h}_N \, d\Gamma$$

Derivation and implementation of GLS. We impose that stabilization term is element-by-element weighted least-squares formulation of the differential equation. So the stabilization introduced by GLS is as follows:

$$\sum_{e}\int_{\Omega}(\boldsymbol{a}\cdot\boldsymbol{\nabla}\rho^{h}-\boldsymbol{\nabla}\cdot(\nu\boldsymbol{\nabla}\rho^{h})+\sigma\cdot\rho^{h})\tau((\boldsymbol{a}\cdot\boldsymbol{\nabla}\rho^{h})-\boldsymbol{\nabla}\cdot(\nu\boldsymbol{\nabla}\rho^{h})+\sigma\cdot\rho^{h}-\mathrm{s})\mathrm{d}\Omega$$

After addition of stabilization term to the equation, we have obtained:

$$\int_{\Omega}\omega^{h}(\mathbf{a}\cdot\boldsymbol{\nabla}\rho^{h})\mathrm{d}\Omega+\int_{\Omega}\boldsymbol{\nabla}\omega^{h}\cdot\left(\nu\boldsymbol{\nabla}\rho^{h}\right)\mathrm{d}\Omega+\int_{\Omega}\omega^{h}\sigma\cdot\rho^{h}\,\mathrm{d}\Omega+\sum_{e}\int_{\Omega}\left(\boldsymbol{a}\cdot\boldsymbol{\nabla}\rho^{h}-\boldsymbol{\nabla}\cdot\left(\nu\boldsymbol{\nabla}\rho^{h}\right)+\sigma\cdot\rho^{h}\right)\cdot$$

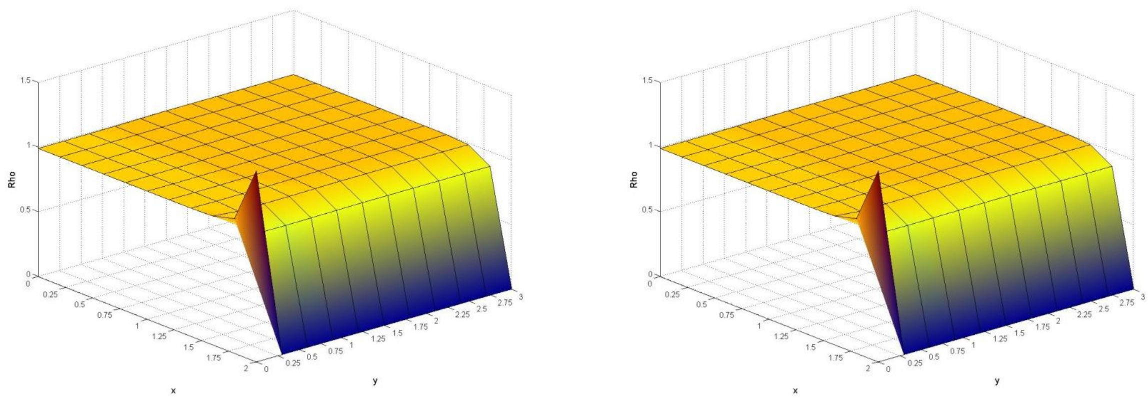
$$\cdot\,\tau(\left(\boldsymbol{a}\cdot\boldsymbol{\nabla}\rho^{h}\right)-\boldsymbol{\nabla}\cdot\left(\nu\boldsymbol{\nabla}\rho^{h}\right)+\sigma\rho^{h})\mathrm{d}\Omega=\int_{\Omega}\omega^{h}\mathrm{s}\,\mathrm{d}\Omega+\sum_{e}\int_{\Omega}(\boldsymbol{a}\cdot\boldsymbol{\nabla}\rho^{h})\tau\mathrm{s}\mathrm{d}\Omega+\int_{\Gamma_{\mathrm{N}}}\omega^{h}{\rho^{h}}_{\mathrm{N}}\,\mathrm{d}\Gamma$$

If the elements are linear $\boldsymbol{\nabla}\cdot\left(\nu\boldsymbol{\nabla}\rho^{h}\right)=0$ both for SUPG and GLS.

Apart from the modification of the code to include SUPG and GLS FEM system, the stabilization parameter definition was modified.

$$\tau=\frac{h}{2a}(1+\frac{9}{Pe^{2}}+\left(\frac{h}{2a}\cdot\sigma\right)^{2})^{-1/2}$$

This is fourth-order accurate formula, which takes into account reaction.



(a) **(b)**

Figure 5. Solution using SUPG (a) and GLS (b) stabilization technique

Comparing to Galerkin space discretization (Figure 1.) this method gives stable solution without spurious oscillations. As the source term σ is small the solution of SUPG and GLS is nearly the same.

Part D. Boundary conditions in the code were changed to impose $\rho=1$ in Γ_4, $\rho=2$ in Γ_2, see Figure 6. Then Neumann boundary conditions were imposed in Γ_2, see Figure 7.

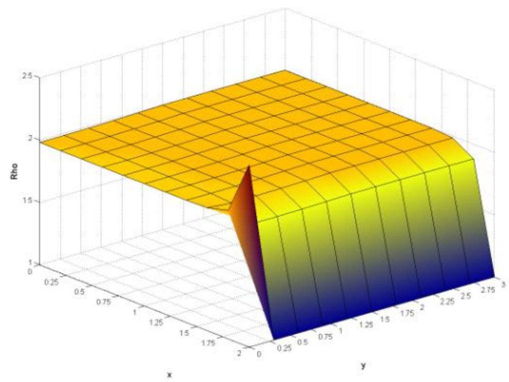


Figure 6. Solution using SUPG stabilization technique with $\rho = 1$ in Γ_4, $\rho = 2$ in Γ_2

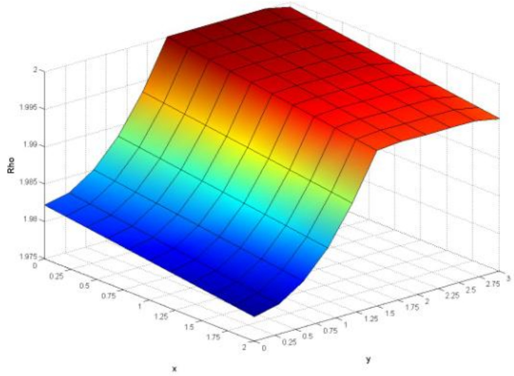


Figure 7. Solution using SUPG stabilization technique with $\rho = 1$ in Γ_4 and Neumann boundary conditions in Γ_2

Exercise 2

Part A. The SUPG formulation is used for space disretization. To develop expression for Crank-Nicolson method we start from the strong form of the problem

$$\mathbf{a} \cdot \boldsymbol{\nabla}\rho - \boldsymbol{\nabla} \cdot (\nu \boldsymbol{\nabla}\rho) + \sigma \cdot \rho = s \quad \text{in } \Omega$$

The Taylor expansion, neglecting third order and higher terms yields:

$$\rho(t^{n+1}) = \rho(t^n) + \Delta t \rho_t(t^n) + \frac{1}{2}\Delta t^2 \rho_{tt}(t^n) + O(\Delta t^3)$$

From the strong form we can obtain time derivatives up to degree, which we are interested in.

$$\rho_t^n = s - \mathbf{a} \cdot \boldsymbol{\nabla}\rho + \boldsymbol{\nabla} \cdot (\nu \boldsymbol{\nabla}\rho) - \sigma \cdot \rho$$

$$\rho_{tt}^n = s_t - \mathbf{a} \cdot \boldsymbol{\nabla}\rho_t + \boldsymbol{\nabla} \cdot (\nu \boldsymbol{\nabla}\rho_t) - \sigma \cdot \rho_t$$

The general formula for θ methods:

$$\frac{\rho(t^{n+1}) - \rho(t^n)}{\Delta t} = \theta \rho_t(t^{n+1}) + (1-\theta)\rho_t(t^n) + O(\left(\frac{1}{2} - \theta\right)\Delta t, \Delta t^2)$$

For our purposes it is convenient to represent this equation in incremental form:

$$\frac{\Delta\rho}{\Delta t} - \theta \Delta\rho_t = \rho_t^n$$

Here $\Delta\rho = \rho(t^{n+1}) - \rho(t^n)$ and $\Delta\rho_t = \rho_t(t^{n+1}) - \rho_t(t^n)$

After inserting time derivatives in the incremental equation we have obtained time-discrete scheme of the problem:

$$\frac{\Delta\rho}{\Delta t} - \theta(\mathbf{a}\cdot\nabla + \nabla\cdot(\nu\nabla) - \sigma)\Delta\rho = \theta s^{n+1} + (1-\theta)s^n - (\mathbf{a}\cdot\nabla + \nabla\cdot(\nu\nabla) - \sigma)\rho^n$$

Crank-Nicolson method corresponds to $\theta = \frac{1}{2}$. Therefore the final expression is as follows:

$$\frac{\Delta\rho}{\Delta t} - \frac{1}{2}(\mathbf{a}\cdot\nabla + \nabla\cdot(\nu\nabla) - \sigma)\Delta\rho = \frac{1}{2}(s^{n+1} + s^n) - (\mathbf{a}\cdot\nabla + \nabla\cdot(\nu\nabla) - \sigma)\rho^n$$

Part B. The code was modified and used to solve the problem with the following parameters: convective term$a(x,y) = (-x,-y)$, diffusion parameter $\nu = 0.04$, reaction term $\sigma = 0.001$, 40 elements in each direction, homogeneous Dirichlet boundary conditions $\rho = 0$ in Γ_4, $\rho = 1$ in Γ_2. Total time of simulation is 4 seconds, 160 time steps are used.

The proposed initial solution $x_0(x,0) = x(2-x)$ was used and results were obtained. Crank-Nicolson and Padé give the same result, see Figure 8.

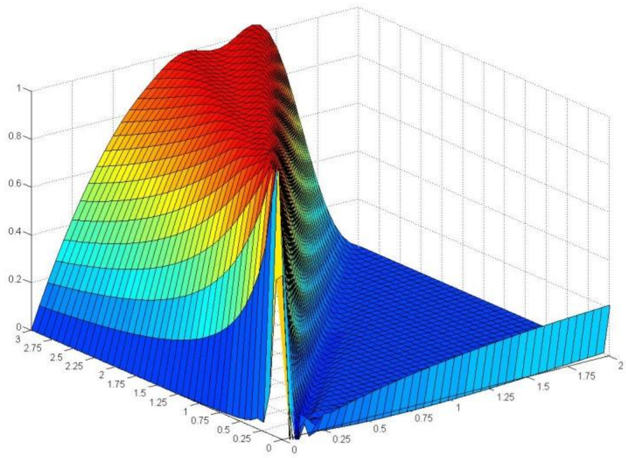
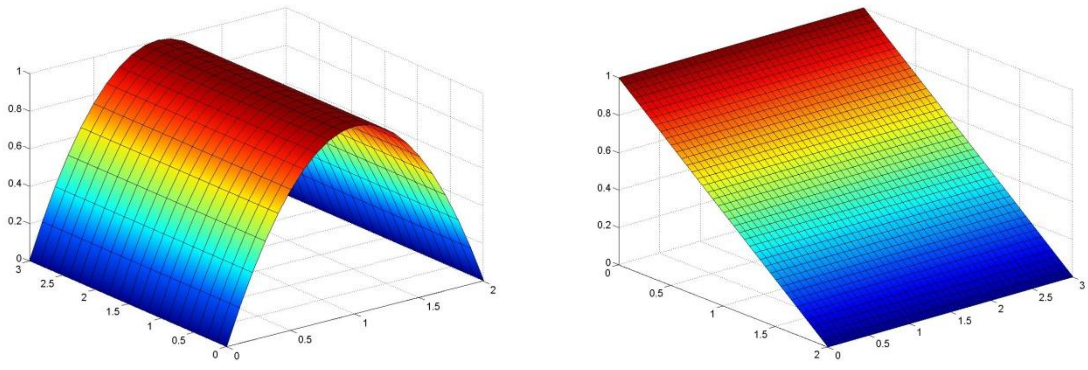


Figure 8. Solution using SUPG for space and Crank-Nicolson and Padé for time

It is obvious that the proposed initial solution does not conform with initial solution at the boundary Γ_4. Therefore it was decided to change the initial solution to the one,which conforms with given boundary conditions. The new initial condition is $x_0(x,0) = 1 - \frac{x}{2}$, see Figure 9.

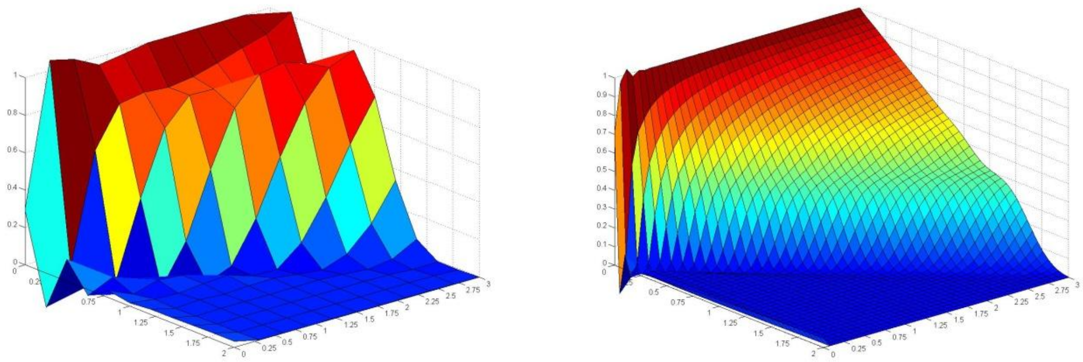


(a) **(b)**

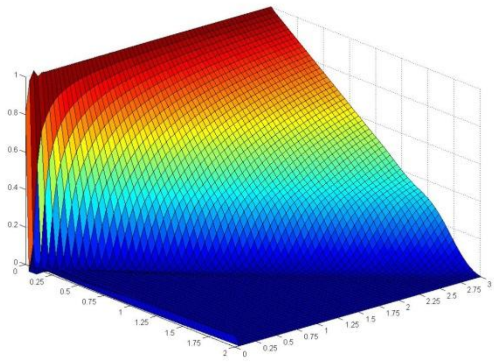
Figure 9. Proposed nonconforming initial condition **(a)** and new conforming initial condition **(b)**

Solution using Padé approximation is more computationally intensive and takes more time. Therefore Crank-Nicolson method is used in Part C to study discretization sensitivity.

Part C. The time frame of $t \in [0,4]$ is considered, so that transient behaviour could be fully analyzed. Mesh discretization sensitivity for the case was studied, see Figure 10.



(a) **(b)**

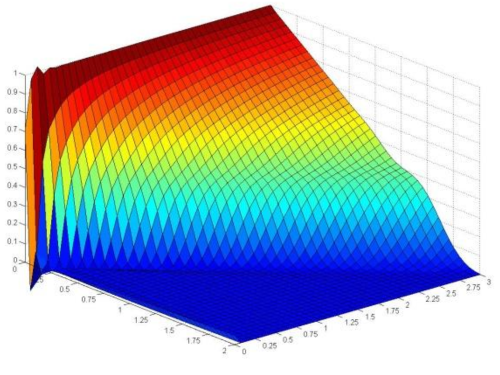


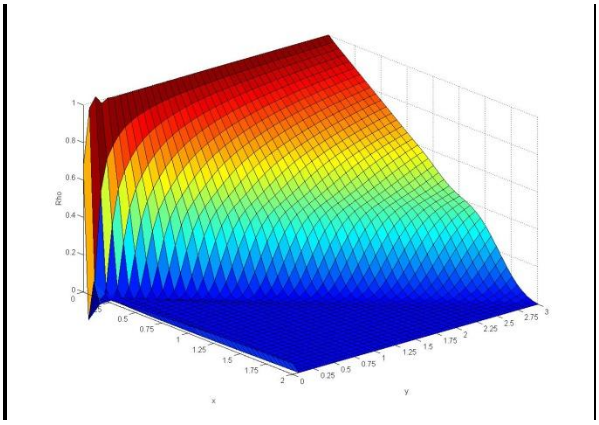
(c)

Figure 10. Solutions for 10 elements **(a)**, 40 elements **(b),** 60 elements **(c)**

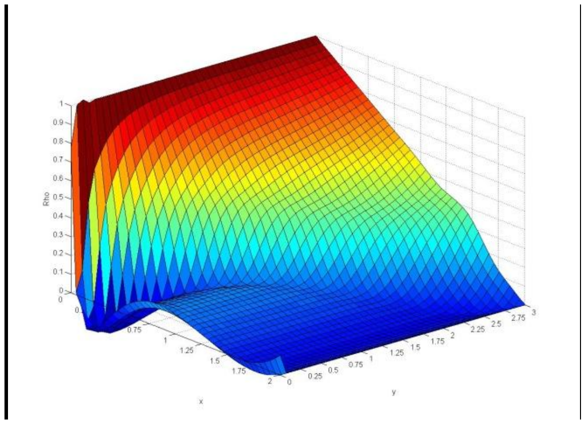
The 10-element mesh has some instabilities and lacks accuracy in general. The solution is converged, while the mesh is refined.

Then the time discretization sensitivity was studied at Crank-Nicolson scheme, see Figure 11.





(a) **(b)**

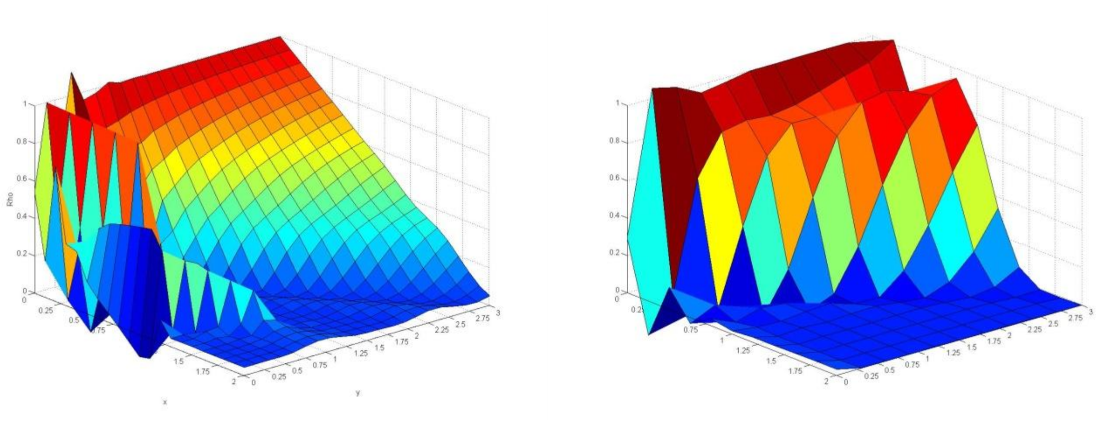


(c)

Figure 11. Solutions for 110 time steps (Courant number: 2.6) **(a)**, 250 time steps (Courant number: 1.15) **(b),** 3 time steps **(**Courant number: 96.1480) **(c)**

As Crank-Nicolson scheme is semi-implicit, it can stand very small time steps. Even when 3 time steps are done, only small spurious oscillations appears.

Part D. The code was modified to study quadratic elements with SUPG space discretizations and parameters from Part C, see Figure 12.



(a) **(b)**

Figure 12 Solution using quadratic elements **(a)**, solutions using linear elements **(b)**

The stabilization, which is used, is not optimal for quadratic elements. Therefore an optimal parameter should be found practically to obtain good results.

References

1. J.Donea, A.Huerta Finite Element Methods for Flow Problems, Wiley, 2003