Finite Elements in Fluids

Final Exam

Name- Prakhar Rastogi

Master in Computational Mechanics

1. (a) $Q_2 Q_1$ finite elements satisfies LBB condition

$$\begin{pmatrix} K & G \\ G^T & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ h \end{pmatrix} \rightarrow \text{Partitioned form}$$

Since LBB condition is satisfied, it means

$$\dim Q^h \leq \dim V^h \quad , \quad p^h \in Q^h, \; v \in V^h$$

and $\ker G = \{0\}$, which is an important condition for symmetric pressure matrix $(G^T K^{-1} G)$ to be positive definite.

Also, the partitioned matrix is now non-singular, therefore solutions are unique.

∴ This pair of finite element spaces suitable to discretise equations.

(b) Stokes problem $L(v,p) = F$ with

$$L(v,p) = \begin{bmatrix} -\nu \nabla^2 v + \nabla p \\ \nabla \cdot v \end{bmatrix} \qquad F = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

GLS stabilisation term added: $\sum_e \int_{\Omega_e} \tau \, L(w,q) \cdot (L(v,p) - F) \, d\Omega$

Since, GLS terms added are ~~from~~ emanated from least-square forms, so momentum and continuity eqs. are modified.

Also, the weak form acts only on interiors of element due to 2^{nd} spatial derivatives.

- Further, Partitioned Matrix contains non-zero diagonal term due ~~to~~ to $(\nabla q^h, \nabla p^h)$, we have a stabilised pressure field.
- Also, elements with equal order interpolations are stable by adding GLS term.
- $\tau = \alpha \frac{h_e^2}{4\nu}$ → Stabilisation parameter.
- Therefore, GLS method is suitable.

(c) Order of convergence of velocity $= p+1$
Pressure has order $= p$
For each element ~~[illegible]~~ $\dim Q \leq \dim V$

~~d.o.f~~ global size $= 8 \times 8$

(d) ~~p~~ Size of global $= 8 \times 8$
" of local $= 3 \times 3$

(e) 1. First, we write given domain into broken domain of system of 1st order eq.

2. Second order eq. for local converted to first order.

3. Solve the local problem and introduce hybrid variable. $\hat{u}$

Add: $p_e = \frac{1}{|\Omega_e|} \langle p_e, 1 \rangle_{\partial\Omega_e}$

4. Check $\langle \hat{u} \cdot n_e, 1 \rangle_{\partial\Omega_e \setminus \Gamma_D} + \langle u_D \cdot n_e, 1 \rangle_{\partial\Omega_e \cap \Gamma_D} = 0$

5. Using jump condition and B.C., we solve global problem.

2. $$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \nabla^2 u + (u\cdot\nabla)u + \sigma u + \nabla p = f & \text{in } \Omega,\ t>0 \quad -①(a) \\ \nabla\cdot u = 0 & \text{in } \Omega,\ t>0 \quad -①(b) \\ u = 0 & \text{on } \partial\Omega,\ t>0 \quad -①(c) \\ u = u_0 & \text{in } \Omega,\ t=0 \quad -①(d) \end{cases}$$

(a) $K(u) = -\nu\nabla^2 u$

$C(u) = (u\cdot\nabla)u$

For Crank-Nicolson $\theta = 1/2$

$$\therefore \frac{u^{n+1}-u^n}{\Delta t} + C(u) + K(u^{n+\frac{1}{2}}) + \sigma u^{n+\frac{1}{2}} + \nabla p^{n+\frac{1}{2}} = f^{n+\frac{1}{2}} \quad ——②$$

and

$$\nabla\cdot u^{n+\frac{1}{2}} = 0$$

where $\Delta t = t^{n+1} - t^n$ $\qquad f^n \approx f(t^n)$

$$f^{n+\frac{1}{2}} = \frac{1}{2}f^{n+1} + \frac{1}{2}f^n$$

Since p has no time derivative, it can be calculated at any time,

$\therefore$ ② becomes

$$\frac{u^{n+1}-u^n}{\Delta t} + C(u) + K(u^{n+\frac{1}{2}}) + \sigma u^{n+\frac{1}{2}} + \nabla p^{n+1} = f^{n+\frac{1}{2}}$$

and

$\nabla\cdot u^{n+1} = 0$ (Incompressibility constraint can be evaluated at any time)

$C(u) = [(u\cdot\nabla)u]^{n+\frac{1}{2}}$ (Implicit)

we consider $C(u)$ as implicit.

(b) $\frac{\partial u}{\partial t} = \frac{u^{n+1} - u^n}{\Delta t} = u_t$

Weak Form

$$(w, u_t) + a(w, u^{n+\frac{1}{2}}) + c(u^{n+\frac{1}{2}}, w, u^{n+\frac{1}{2}}) + b(w, p^{n+1})$$
$$+ d(w, u^{n+\frac{1}{2}}) = (w, f^{n+\frac{1}{2}})$$
$$b(u^{n+1}, q) = 0$$

$w \in V$; $q \in Q$; $u(x,t) \in S \times]0, T[$

$p(x,t) \in Q \times]0, T[$

Now u and p approx. as $u^h \in V^h$, $p^h \in Q^h$

$(w^h, q^h) \in V^h \times Q^h$

$$(w^h, u_t^h) + a(w^h, u^{h\,n+\frac{1}{2}}) + c(u^{h\,(n+\frac{1}{2})}, w^h, u^{h\,n+\frac{1}{2}})$$
$$+ b(w^h, p^{h\,n+1}) + d(w^h, u^{h\,n+\frac{1}{2}}) = (w^h, f^{h\,n+\frac{1}{2}})$$
$$b(u^{h\,n+1}, q^h) = 0$$

Now:

$$a(w, u^{n+\frac{1}{2}}) = \int (\nabla w) : (\nu \nabla u^{n+\frac{1}{2}})\, d\Omega$$

$$b(u^{n+1}, q) = \int_\Omega q\, \nabla \cdot v\, d\Omega$$

$$c(w, u^{n+\frac{1}{2}}, a) = \int w \cdot (v^{n+\frac{1}{2}} \cdot \nabla)\, v^{n+\frac{1}{2}}\, d\Omega$$

$$d(w, u^{n+\frac{1}{2}}) = \int \sigma\, w \cdot u^{n+\frac{1}{2}}\, d\Omega$$

$$(w, f^{n+\frac{1}{2}}) = \int w \cdot f^{n+\frac{1}{2}}\, d\Omega$$

(c) After discritisation we have.

$$M \underbrace{\dot{u}(t)}_{u^n} + (K + C(u(t)) + D)\, u^{n+\frac{1}{2}} + G^T p^{n+1} = 0$$

$$G u^{n+1} = 0$$

$$\cancel{\begin{bmatrix} K + C(u) + D & G^T & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} u^{n+\frac{1}{2}} \\ p^{n+1} \\ u^{n+1} \end{bmatrix} = 0}$$

$$M u^{n+1} - M u^{n} + \Delta t\, (K + C(u) + D)\, u^{n+1/2} + \Delta t\, G^T p^{n+1} = 0$$

$$G u^{n+1} = 0$$

$$\begin{bmatrix} M & -M & \Delta t (K + C(u) + D) & \Delta t\, G^T \\ G & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ u^{n} \\ u^{n+\frac{1}{2}} \\ p^{n+1} \end{bmatrix} = 0$$

(d) Algorithm to solve non-linear terms.

Picard Method

- $A(x)\, x = b(x)$ (Applicable to systems)
- Loop
  - initial value $x = x^0$
  - Run loop until x^{k+1} convergence.

$A(x^k)\, x^{k+1} = b(x^k)$ → At each iteration linear system is solved.

- Thus at last we have partioned matrix and we can cal. the required Matrices

Newton Method

- Non-linear system : $r(x) = 0$
- Loop
  - initial value $x = x^0$
  - Solve sys of eq^n

$$J(x^k)\, \Delta x^{k+1} = -r(x^k)$$

  - update sol

$$x^{k+1} = x^k + \Delta x^k$$

Jacobian Matrix which is computed in each step of loop

$$[J(x)]_{ij} = \frac{\partial r_i}{\partial x_j}(x)$$

$$r(x) = Ax(x) - b(x)$$

$$J(x) = A(x) + \frac{\partial A(x)}{\partial x} x - \frac{\partial b(x)}{\partial x}$$

- But ideal case, will be if we can combine Picard Method ~~to~~ with Newton-Rapson.

Picard Method will take you closer to the stable sol^n. After, we reach a approx. close to solution, then, with help of Newton Rapson we reach the sol^n quickly.

(e) Yes, methods ~~having~~ are behaving as expected.

- For low Re = 100, Picard takes more no. of steps than Newton Raphson, i.e., N-R method graph converges quickly to soln

- However for high Re No, i.e., Re = 1000 Picard produces stable soln while Newton Raphson Method becomes unstable

To explain this, let us consider cavity problem.

For ~~the~~ low Re No, i.e., Re = 100 only 1 cavity region.

while For high Re No, i.e., Re = 1000, more than one cavity which means we having more than one minima. Therefore, Newton Raphson method is unable to identify in which direction ~~to~~ to go.