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FEF Final Exam

Masters in Numerical Methods of Engineering

Q2

$$\frac{\partial \bar{u}}{\partial t} - \nu \nabla^2 \bar{u} + (\bar{u}\cdot\nabla)\bar{u} + \sigma \bar{u} + \nabla p = f \quad \text{in } \Omega \quad t>0$$

$$\bar{\nabla}\cdot\bar{u} = 0 \qquad \text{in } \Omega \; t>0$$

$$\bar{u} = 0 \qquad \text{on } \partial\Omega \quad t>0$$

$$u = u_0 \qquad \text{in } \Omega \; t=0$$

Ex.

a) write the time discretization of the problem using Crank-Nicolson method

→

$$\cancel{u_t = f - (K + C(u))u}$$

$$u_t = f - \nabla p - \nu \nabla^2 \bar{u} \mp (\bar{u}\cdot\nabla)\bar{u} - \sigma \bar{u}$$

The time discretization using θ methods is given by

$$\frac{\Delta u}{\Delta t} - \theta \Delta u_t = u_t^n$$

n = time step

$\theta = \frac{1}{2}$ for crank nicolson scheme

$$\therefore \quad \cancel{\frac{\Delta u}{\Delta t} - \theta\left(\nu \nabla^2 u + (\bar{u}\cdot\nabla)u + \sigma \bar{u}\right) = f - \nabla p - \nu \nabla^2 \bar{u} \mp (u\cdot\nabla\bar{u})}$$

Let us rewrite the equation in a simpler form

$$\bar{u}_t + \underset{(\bar{v}\cdot\bar{\nabla})\bar{v}}{C(\bar{u})} + \underset{-\nu\nabla^2 v}{K(\bar{u})} + \underset{\sigma u}{S(\bar{u})} + \nabla p = f$$

So the classic one step scheme yield

$$\frac{u^{n+1} - u^n}{\Delta t} + C(u) + K(u^{n+\theta}) + S(\bar{u}) + \nabla p^{n+\theta} = f^{n+\theta}$$

$$\nabla \cdot v^{n+\theta} = 0$$

$\Delta t = t^{n+1} - t^n$ $\quad f^{n+\theta} = \theta f^{n+1} + (1-\theta) f^n$

for CN $\dfrac{f^{n+1} + f^n}{2}$

Now for CN $\theta = \frac{1}{2}$ we get

$$\frac{\bar{u}^{n+1} - \bar{u}^n}{\Delta t} + C(\bar{u}) + K(v^{n+1/2}) + \nabla p^{n+1} + S(\bar{u}^{n+1}) = f^{n+1/2}$$

and $\nabla \cdot \bar{u}^{n+1} = 0$

& $C(\bar{v}) = ((\bar{u} \cdot \nabla)\bar{u})^{n+1/2}$

Q1b Derive the weak form of the problem stated in a

$$\frac{\partial \bar{u}}{\partial t} - \nu \nabla^2 \bar{u} + (\bar{u} \cdot \bar{\nabla})\bar{u} + \sigma \bar{u} + \nabla p = f \quad \text{in } \Omega \; t > 0$$

$$\bar{\nabla} \cdot \bar{u} = 0 \quad \text{in } \Omega \; t \geq 0$$

Let us consider the weighted residual approach,

we multiply by a test function $\bar{w}$ such that $\bar{w} \in V$ for the momentum equation & $q \in Q$ for the pressure equation.

$$\int_\Omega w \frac{\partial \bar{u}}{\partial t} + -\int_\Omega \bar{w}\cdot(\nu \nabla \cdot \bar{u})\, d\Omega + \int_\Omega w\,(\bar{u}\cdot\bar{\nabla})\bar{u} + \int \bar{w}\cdot\sigma\bar{u} + \oint w\cdot\nabla p\, d\Omega$$

$$= \int_\Omega w f\, d\Omega$$

$$\int_\Omega q\,\bar{\nabla}\cdot\bar{u}\, d\Omega = 0$$

this takes the form, after performing integration by parts,

$$(\bar{w}, \bar{u}_t) + a(\bar{w}, \bar{u}) + c(\bar{u} : \bar{w}, \bar{u}) + b(\bar{w}, p) + (\bar{w}, \sigma\bar{u}) = (\bar{w}, b) \quad \forall w \in V$$

$$b(\bar{v}, \bar{q}) = 0 \qquad \forall q \in Q$$

where

$$a(\bar{w}, \bar{u}) = \int_\Omega \bar{\nabla}\bar{w} : (\nu \nabla\bar{u})\, d\Omega \qquad \& \qquad b(\bar{u}, \bar{q}) = \int_\Omega q\,(\bar{\nabla}\cdot\bar{u}) = 0$$

$$c(\bar{u}; w, \bar{u}) = \int_\Omega w(\bar{u}\cdot\nabla u)\, d\Omega$$

Now $\bar{Q}$ & $\bar{V}$ are defined as the following spaces

$$\bar{Q} = \{p \in L_2(\Omega)\}$$
$$\bar{V} = \{\bar{u} \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$$

~~Q2c Discretize the weak form~~ and write the system of equations.

Now we finally write the galerkin formulation with the theta family of methods

$$\left(w, \frac{\Delta u}{\Delta t}\right) + \theta\left(c(\bar{u}; \bar{w}\,\Delta\bar{u}) + a(\bar{w}, \Delta\bar{u}) + b(w, p) + (\bar{w}, \sigma\Delta\bar{u})\right) = $$
$$- \left[c(\bar{u}; \bar{w}, \bar{u}^n) + a(\bar{w}, u^n) + (w, \sigma u^n) + b(\bar{w}, p)\right]$$
$$+ (w, \theta f^{n+1} + (1-\theta) f^n)$$

$$\& \int q \cdot (D \cdot \bar{u}^{n+1})\, d\Omega = 0 \quad q \in \mathcal{Q}$$

~~Now~~

~~$\S\S$~~ Discretization of the weak form

~~$\S\S$~~ C

$$\bar{u} \simeq \bar{u}^h = \sum_{j=1}^{N} \bar{u}_j \bar{N}_j \qquad p \simeq p^h = \sum_{j=1}^{n} \hat{N}_j \, p_j \qquad \bar{w} = w^h = \sum_{j=1}^{N} \bar{N}_i \bar{w}_i$$

$$\& \; q \simeq q^h \sum_{j=1}^{N} \hat{N}_i q$$

Now putting these discretized form in the previous equation we obtain the individual terms to be as follows

$$\int \frac{\bar{w} \cdot \bar{u}}{\Delta t} = \int \frac{\bar{N}^T \bar{N} u \, d\Omega}{\Delta t} = \bar{M}$$

$$\int (D\bar{w}) : Dv = \int (\text{grad}\, \bar{N})^T (\text{grad}\, N) \; w.r = \bar{K}(u)$$

$$\int \sigma \cancel{\ldots} \cdot (\bar{w}, \sigma \bar{u}) = \int \sigma \, N^T N \, \bar{u} \, d\Omega = \bar{R}(u)$$

$$\int \bar{w} \cdot (\bar{u} \cdot D)\bar{u}) \, d\Omega = \int (\bar{N}^T)(\bar{N} u^*)(\text{grad}\, \bar{N}) \, u \, d\Omega = \bar{C}(\bar{u})$$

$$\int \text{tr}(D \cdot \bar{w}) \; P \, d\Omega = \int [1 \; 0 \; 0 \; 1] (\text{grad}\, N^T) \hat{N} \, P \, d\Omega = G^T P$$

$$\& \int_\Omega q \, (D \cdot \bar{u}) \, d\Omega = \int \hat{N}^T [1 \; 0 \; 0 \; 1] (\text{grad}\, N) \; \bar{u} \, d\Omega = \bar{G} \bar{u}$$

(*) → Matrices M, K, u, k will vary was u^{n+1} or u^n depending on where they are placed

∴ These are the vectors & matrices that will be present

~~Q2d~~

Now after discretization & using the matrix form we get

$$\cancel{\frac{M + \theta K + C}{\Delta t}}$$

$$\cancel{\frac{\Delta u}{\Delta t} - \theta \frac{\left((K+C)\Delta u + G\Delta p + R\right)}{M}} = f - (K + C^n) u$$

$$\frac{\Delta u}{\Delta t} - \theta\left(\frac{(K + C^n + R)\Delta u + G\Delta p}{M}\right) = \frac{f - (K + C^n + R)u^n - Gp^n}{M}$$

we finally obtain

$$(M + \theta \Delta t (K + C^n + R))\Delta u + \theta \Delta t G \Delta p = \Delta t (f - (K + C^n + R)u^n - Gp^n)$$

$$G^T \Delta u = 0$$

in the matrix form

$$\begin{bmatrix} M + \theta \Delta t (K + C^n + R) & \Delta t \theta G \\ G^T & 0 \end{bmatrix} \begin{Bmatrix} \Delta u \\ \Delta t \end{Bmatrix} = \begin{bmatrix} \overset{M/\Delta t \; u^n}{\cancel{\Delta t (f - (K + C^n + R)u^n} - Gp^n} \\ 0 \end{bmatrix}$$

Q2d Propose an algorithm to solve the non-linear problem, detailing its steps

→ On each time step we need to solve a nonlinear equation we can use Picard or Newton Raphson Method to do the same.

Let us look at Picard method.

We have a a system $A(x)x = b(x)$

So ~~we can say~~

and this is an iterative algorithm, we take an initial guess for x^0 in our case for pressure & velocity

& we solve

$$A(x^k)x^{k+1} = b(x^k) \quad \text{for each iteration}$$

to be specific since we have a semi implicit method in ~~Newton Ra~~ Crank Nicolson we will solve the ~~system~~ & the matrix at a previous time step

$$\therefore \begin{bmatrix} M + \Delta t\theta(K + R + C(v^n)) & G^T \\ G & 0 \end{bmatrix} \begin{Bmatrix} \bar{u}^{n+1} \\ p \end{Bmatrix} = \begin{Bmatrix} M/\Delta t\, \bar{u}^n \\ 0 \end{Bmatrix}$$

32e $\sigma = 0$

In figure 1 (Re100) we can see that the Picard's method converges linearly while the Newton Raphson converges quadratically. (only 5 iterations)

In figure 2 (Re = 1000) Picard's method still converges linearly but Newton Raphson does not converge

We can say that the figures show expected behaviours

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Since Newton Raphson has a ~~bit~~ faster convergence, the ~~so~~ steps taken towards the solution would be larger. ~~Thee~~ As we approach higher Reynolds numbers the instability in the system would also increase. Thus Newton Raphson does not converge for Re = 1000. It can also be observed that even Picard's Method will not converge for even more Re ~ 5000.

Problem 1

Stokes problem

$$-\nu \nabla^2 \bar{v} + \nabla \bar{p} = b \quad \text{in } \Omega$$

$$\bar{\nabla} \bar{v} = 0 \quad \text{in } \Omega$$

$$v = 0 \quad = \text{on } \partial\Omega$$

Q1a A ~~The~~ Q2Q1 elements ~~satis~~ satisfy the LBB condition which is $\dim Q^h \le \dim V^h$

Q2Q1 element has continuous biquadratic velocity & continuous bilinear pressure.

Therefore this ~~element~~ pair of FE space is suitable for discretization

Q1b Since a Q2 Q1 element already satisfies the LBB condition we do not need any kind of ~~satis~~ stabilization to solve the problem. The stokes problem is expected to converge

Q1c For HDG we see that

$$\|e_u\|_{L_2(\Omega)} \le Ch^{p+1} |u|_{H^{p+1}(\Omega)}$$

& $$\|e_q\|_{L_2}(\Omega) \le Ch^{p+1} |q|_{H^{p+1}}(\Omega)$$

∴ if ~~pressure~~ is.

If velocity is using polynomial upto degree six we can have pressure using the same order of polynomial as well

Q1c Since it is a DG problem & velocity has been approximated by the order 6 we need pressure to be approximated such that it satisfies the LBB condition

such that
$$\dim Q^h \leq \dim V^h$$

$\therefore$ $\underline{P \leq 6}$ for pressure

For this problem we have 15 velocity nodes & pressure nodes on each of the row $\therefore$ we will have $15 \times 15 \times 2$ ~~$1 \times 11 \times 2$~~ for velocity variables & $= 450$

$15 \times 15 \times 1$ ~~$11 \times 11 \times 1$~~ pressure variables $= 225$

Q1~~e~~

but subtracting ~~12×4~~ $14 \times 4 \times 2$ velocity nodes at the boundary

$14 \times 4 \times 1$ pressure nodes at the boundary we get

&

338 velocity variables + 169 pressure variables

$\therefore$ The reduced system is $\underline{507 \times 507}$

Q1d Now the degree of freedom for each variable in a DG is given by

$$u(\xi) \simeq u^h(\xi) = \sum_{i=1}^{nen} u_j N_j(\xi)$$

$$q(\xi) \simeq q^h(\xi) = \sum_{i=1}^{gen} q_j N_j(\xi)$$

$$\hat{u}(\xi) = u^h(\xi) = \sum_{i=1}^{fn} \hat{u}_j \hat{N}_j(\xi)$$

fn = no of nodes on faces

We will have 48 nodes on faces, 6 per each face $\times$ 8 faces

So the global system will be of $48 \times 2 + 48 = 144 \times 144$ nodes

~~Goto~~ continued later

Q1d contd

Each to element has 10 internal nodes ∴ the local system will be of the order 30x30

So for the global problems we will need to adjust for the values of pressure on the subdomain boundary.

Q1e

The steps required to solve the problem are.

1) ~~B~~ compute and do assembly of block matrices for local & global problems

2) Solve the global problem

3) solve the local problem for each element one by one

4) Compute the preprocessed solution ~~of the~~

after the discretization of the weak form the local problem becomes

$$\begin{bmatrix} A_{LL} & A_{Lu} & 0 & 0 \\ A_{Lu}^T & A_{uu} & A_{pu}^T & 0 \\ 0 & A_{pu} & 0 & a^T_{pp} \\ 0 & 0 & a_{pp} & 0 \end{bmatrix} \begin{bmatrix} L_e \\ u_e \\ p_e \\ \xi_e \end{bmatrix} = \begin{bmatrix} f_L \\ f_u \\ f_p \\ \beta \end{bmatrix}_e + \begin{bmatrix} A_{L\hat{u}} \\ A_{u\hat{u}} \\ A_{p\hat{u}} \\ 01 \end{bmatrix}_e \hat{u}_e$$

for all elements $e = 1 \cdots n_{el}$

The global problem takes the form

$$\begin{bmatrix} \hat{K} & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} \hat{f}_u \\ \hat{f}_p \end{bmatrix}$$