Burger's equation is spatially discretized using the Galerkin method resulting in the following system of equations:

$$\mathbf{M}\dot{\mathbf{U}} + \mathbf{C}(\mathbf{U})\mathbf{U} + \epsilon\mathbf{K}\mathbf{U} = 0$$

Equation 1 System of equations to be solved

In order to solve this system, the application of the forward Euler method will produce a linear system of equations. This system could be solved explicitly for each time step. Another approach is to apply the backward Euler method which will produce non-linear system of equations. This due to the dependency of the convection matrix C on U. In order to solve such a system, an iterative scheme could be employed namely the Picard or Newton-Raphson methods.

The following results are obtained for the explicit, Picard and Newton Raphson methods respectively by varying the values of time increment and the viscosity.

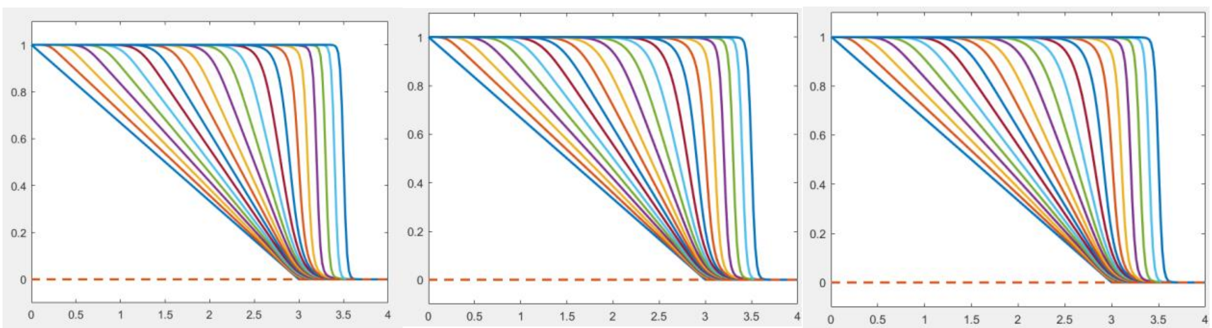

Figure 1 Explicit, Picard, and Newton-Raphson methods (E= 1e-2; dt = 0.005)

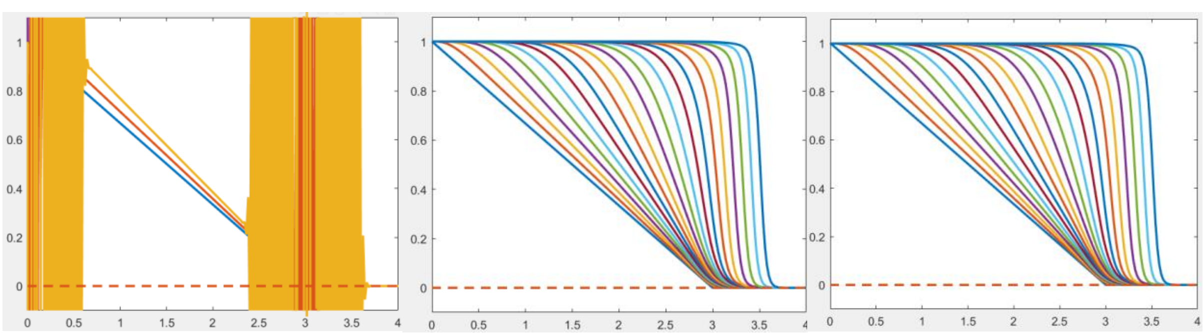

Figure 2 Explicit, Picard, and Newton-Raphson methods (E= 1e-2; dt = 0.05)

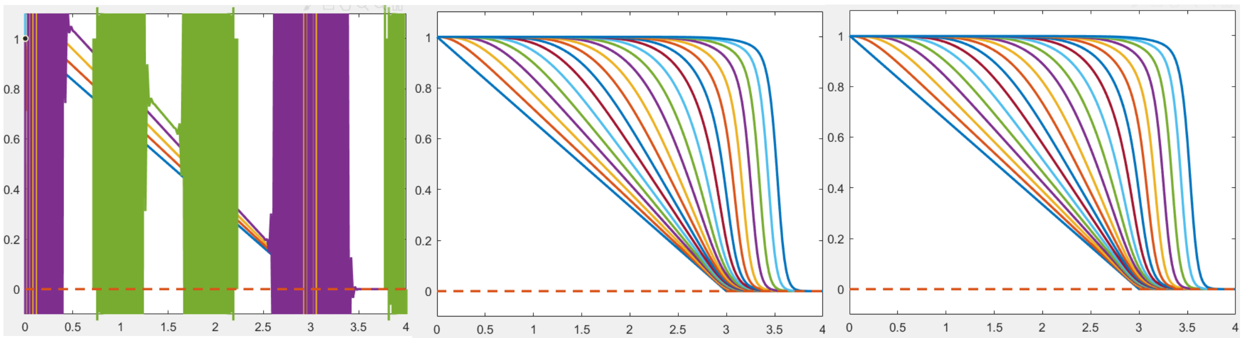

Figure 3 Explicit, Picard, and Newton-Raphson methods (E= 1e-2; dt = 0.1)

For Figure 1 the results of all the methods are stable and they are almost equivalent. However, in Figure 2 and Figure 3 the results of the explicit method show instability while the results of both implicit methods show stability. The difference between these cases is the increase of the time step. This implies that the explicit method doesn't behave well using large time steps. This is due to the fact that the Euler method is first order convergent and it requires small time steps to achieved stability.

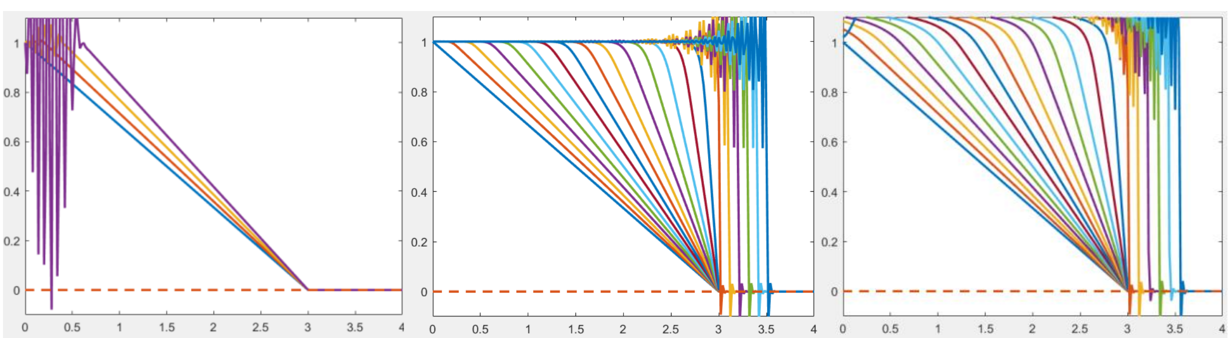

Figure 4 Explicit, Picard, and Newton-Raphson methods (E= 1e-4; dt = 0.005)

In Figure 4, all the methods show instability in the solution. This could be attributed to the domination of the convection term over the diffusion term. With the decrease in viscosity, singularities start to appear which implies the existence of a discontinuity in the solution.

In conclusion, Burger's equation could be solved using implicit or explicit schemes. The implicit methods will need to be solved using iterative methods such as Picard or Newton-Raphson. Which requires higher computational cost compared to the explicit scheme. The main difference between the two approaches is that the explicit method is conditionally stable and requires low time increments to insure stability. The implicit methods are unconditionally stable for any time increment size. However, all the methods show instability when using a low diffusion coefficient.

Newton-Raphson MATLAB modified code

```
for n = 1:nTimeSteps
    %fprintf('\nTime sdtep %d\n', n);
    bccd = [uxa; uxb];
    U0 = U(:,n);
    error_U = 1; k = 0;
    while (error_U > 0.5e-5) && k < 20
        C = computeConvectionMatrix(X,T,U0);
        f = ( M + At*C + E*At*K ) * U0  - M * U(:,n);
        J = M + 2*At*C + E*At*K ;
        U1 = U0 - J^-1 * f ;

        error_U = norm(U1-U0)/norm(U1);
        %fprintf('\t Iteration %d,
error_U=%e\n',k,error_U);
        U0 = U1; k = k+1;
    end
```