Master Of Science in Computational Mechanics

Finite Elements in Fluids

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Assignment 8

(Hybridisable Discontinuous Galerkin-Ex No. 12)

Consider the domain $\Omega = [0,1]^2$ such that $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$ with $\Gamma_D \cap \Gamma_N = \phi$, $\Gamma_D \cap \Gamma_R = \phi$., $\Gamma_N \cap \Gamma_R = \phi$. More precisely, set

$$\Gamma_N := \{(x,y) \in \mathbb{R}^2 \ : \ x = 0\},$$
$$\Gamma_R := \{(x,y) \in \mathbb{R}^2 \ : \ y = 0\},$$
$$\Gamma_D := \partial\Omega \backslash (\Gamma_N \cup \Gamma_R).$$

The following second-order linear scalar partial differential equation is defined

$$\begin{cases} -\nabla \cdot (\kappa \nabla u) = s & \text{in } \Omega, \\ u = u_D & \text{on } \Gamma_D \\ \mathbf{n} \cdot (\kappa \nabla u) = t & \text{on } \Gamma_N \\ \mathbf{n} \cdot (\kappa \nabla u) + \gamma u = g & \text{on } \Gamma_R \end{cases} \tag{1}$$

1 HDG formulation of the problem 1

Strong form over the domain can be written as:

$$\begin{cases} -\nabla \cdot (\kappa \nabla u) = s & \text{in } \Omega_i, \text{ and for } i = 1, ..., n_{el}, \\ u = u_D & \text{on } \Gamma_D \\ \mathbf{n} \cdot (\kappa \nabla u) = t & \text{on } \Gamma_N \\ \mathbf{n} \cdot (\kappa \nabla u) + \gamma u = g & \text{on } \Gamma_R \\ [\![u\mathbf{n}]\!] = \mathbf{0} & \text{on } \Gamma \\ [\![\mathbf{n} \cdot \nabla u]\!] = 0 & \text{on } \Gamma \end{cases}$$

The two last equations introduced correspond to the imposition of the continuity of the primal variable u and the normal fluxes respectively along the internal interface Γ.

The first phase of solving HDG formulation is by, element-by-element problem defined with (q, u) as unknowns defined as:

$$\begin{cases} \nabla \cdot \mathbf{q}_i = s & \text{in } \Omega_i, \\ \mathbf{q}_i + \kappa \nabla u_i = \mathbf{0} & \text{in } \Omega_i, \\ u_i = u_D & \text{on } \partial\Omega_i \cap \Gamma_D, \\ u_i = \hat{u} & \text{on } \partial\Omega_i \backslash \Gamma_D, \end{cases}$$

for $i = 1, .., n_{el}$.
And the second phase, a global problem is defined to determine hybrid variable $\hat{u}$. The imposition of the so-called transmission conditions:

$$\begin{cases} [\![\mathbf{n} \cdot \mathbf{q}]\!] = 0 & \text{on } \Gamma \\ \mathbf{n} \cdot \mathbf{q} = -t & \text{on } \Gamma_N \\ \mathbf{n} \cdot \mathbf{q} = \gamma \hat{u} - g & \text{on } \Gamma_R \end{cases}$$

1.1 Weak form

The weak formulation is:

$$-(\nabla v, \mathbf{q}_i)_{\Omega_i} + \langle v, \mathbf{n}_i \cdot \hat{\mathbf{q}}_i \rangle_{\partial\Omega_i} = (v, s)_{\Omega_i}$$
$$-(\mathbf{w}, \mathbf{q}_i)_{\Omega_i} + \kappa(\nabla \cdot \mathbf{w}, u_i)_{\Omega_i} = \kappa \langle \mathbf{n}_i \cdot \mathbf{w}, u_D \rangle_{\partial\Omega_i \cap \Gamma_D} + \kappa \langle \mathbf{n}_i \cdot \mathbf{w}, \hat{u} \rangle_{\partial\Omega_i \setminus \Gamma_D}$$

where the numerical traces of the fluxes $\hat{\mathbf{q}}_i$ are defined, element-by-element as:

$$\mathbf{n}_i \cdot \hat{\mathbf{q}}_i := \begin{cases} \mathbf{n}_i \cdot \mathbf{q}_i + \tau_i (u_i - u_D) & \text{on } \partial\Omega_i \cap \Gamma_D \\ \mathbf{n}_i \cdot \mathbf{q}_i + \tau_i (u_i - \hat{u}) & \text{elsewhere} \end{cases}$$

with τ_i being a stabilization parameter.

The weak form of the global problem is defined by $\hat{u} \in \mathcal{M}(\Gamma \cup \Gamma_N \cup \Gamma_R)$ for all $\mu \in \mathcal{M}(\Gamma \cup \Gamma_N \cup \Gamma_R)$ such that:

$$\sum_{n=1}^{n_{el}} \langle \mu, \mathbf{n}_i \cdot \hat{\mathbf{q}}_i \rangle_{\partial\Omega_i \setminus \partial\Omega} + \sum_{n=1}^{n_{el}} \langle \mu, \mathbf{n}_i \cdot \hat{\mathbf{q}}_i + t \rangle_{\partial\Omega_i \cap \Gamma_N} + \sum_{n=1}^{n_{el}} \langle \mu, \mathbf{n}_i \cdot \hat{\mathbf{q}}_i + g - \gamma \hat{u} \rangle_{\partial\Omega_i \cap \Gamma_R} = 0$$

$$\sum_{n=1}^{n_{el}} \left\{ \langle \mu, \mathbf{n}_i \cdot \mathbf{q}_i \rangle_{\partial\Omega_i \setminus \Gamma_D} + \langle \mu, \tau_i u_i \rangle_{\partial\Omega_i \setminus \Gamma_D} - \langle \mu, \tau_i \hat{u} \rangle_{\partial\Omega_i \setminus \Gamma_D} - \langle \mu, \gamma \hat{u} \rangle_{\partial\Omega_i \cap \Gamma_R} \right\}$$
$$= - \sum_{n=1}^{n_{el}} \left\{ \langle \mu, g \rangle_{\partial\Omega_i \cap \Gamma_R} + \langle \mu, t \rangle_{\partial\Omega_i \cap \Gamma_N} \right\}$$

1.2 Spatial discretization

The following discrete finite-element spaces are introduced:

$$\begin{aligned} \mathcal{W}^h(\Omega) = \quad & \{\mathbf{w} \in [\mathcal{L}_2(\Omega)]^2;\, w|_{\Omega_i} \in [\mathcal{P}^p(\Omega_i)]^2 \ \forall \Omega_i\} & \subset \mathcal{W}(\Omega) \\ \mathcal{V}^h(\Omega) = \quad & \{v \in [\mathcal{L}_2(\Omega)]^2;\, v|_{\Omega_i} \in \mathcal{P}^p(\Omega_i) \ \forall \Omega_i\} & \subset \mathcal{V}(\Omega) \\ \mathcal{M}^h(S) = \quad & \{\mu \in [\mathcal{L}_2(S)]^2;\, \mu|_{\Gamma_i} \in \mathcal{P}^p(\Gamma_i) \ \forall \Omega_i \subset S \subset \Gamma \cup \partial\Omega\} & \subset \mathcal{M}(S) \end{aligned}$$

The variables u, q, and $\hat{u}$ are approximated as:

$$\mathbf{q} \approx \mathbf{q}^h = \sum_{n=1}^{n_{el}} N_j \mathbf{q}_j \quad \in \mathcal{W}^h$$

$$u \approx u^h = \sum_{n=1}^{n_{el}} N_j u_j \quad \in \mathcal{V}^h$$

$$\hat{u} \approx \hat{u}^h = \sum_{n=1}^{n_{el}} \hat{N}_j \hat{u}_j \quad \in \mathcal{M}^h(\Gamma \cup \Gamma_N \cup \Gamma_R) \text{ or} \mathcal{M}^h(\Gamma)$$

The weak form of the local problem gives rise to the following system of equations for each element:

$$\begin{bmatrix} \mathbf{A_{uu}} & \mathbf{A_{uq}} \\ \kappa \mathbf{A_{uq}}^T & \mathbf{A_{qq}} \end{bmatrix}_i \begin{bmatrix} \mathbf{u_i} \\ \mathbf{q_i} \end{bmatrix} = \begin{bmatrix} \mathbf{f_u} \\ \kappa \mathbf{f_q} \end{bmatrix}_i + \begin{bmatrix} \mathbf{A_{u\hat{u}}} \\ \kappa \mathbf{A_{q\hat{u}}} \end{bmatrix}_i \hat{\mathbf{u}}_i$$

Applying the interpolation

$$\sum_{n=1}^{n_{el}} \left\{ \begin{bmatrix} \mathbf{A}_{\mathbf{u\hat{u}}}^T & \mathbf{A}_{\mathbf{q\hat{u}}}^T \end{bmatrix}_i \begin{bmatrix} \mathbf{u_i} \\ \mathbf{q_i} \end{bmatrix} + \left[\mathbf{A_{\hat{u}\hat{u}}}\right]_i \hat{\mathbf{u}}_i + \left[\mathbf{A}_{\mathbf{\hat{u}\hat{u}}}^R\right]_i \hat{\mathbf{u}}_i \right\} = \sum_{n=1}^{n_{el}} \left\{ \left[\mathbf{f_{\hat{u}}}\right]_i + \left[\mathbf{f}_{\mathbf{\hat{u}}}^R\right]_i \right\}$$

In the above equation the matrices $\mathbf{A}_{\mathbf{\hat{u}\hat{u}}}^R$ and $\mathbf{f}_{\mathbf{\hat{u}}}^R$ are associated to the Robin boundary condition of the problem and are defined as:

$$\mathbf{A}_{\mathbf{\hat{u}\hat{u}}}^R = -\sum_{\partial\Omega_i \cap \Gamma_R} \gamma \sum_{g=1}^{n_{ip}^f} \hat{\mathbf{N}}_{\mathbf{n}}(\xi_{\mathbf{g}}^{\mathbf{f}}) \hat{\mathbf{N}}^T(\xi_{\mathbf{g}}^{\mathbf{f}}) w_g^f$$

$$\mathbf{f}_{\mathbf{\hat{u}}}^R = -\sum_{\partial\Omega_i \cap \Gamma_R} \sum_{g=1}^{n_{ip}^f} \mathbf{N}(\xi_{\mathbf{g}}^{\mathbf{f}}) g(\mathbf{x}(\xi_{\mathbf{g}}^{\mathbf{f}})) w_g^f$$

After substituting the local solution

$$\hat{\mathbf{K}} \hat{\mathbf{u}} = \hat{\mathbf{f}}$$

with

$$\hat{\mathbf{K}} = \mathop{\mathrm{A}}_{i=1}^{n_{el}} \begin{bmatrix} \mathbf{A}_{\mathbf{u\hat{u}}}^T & \mathbf{A}_{\mathbf{q\hat{u}}}^T \end{bmatrix}_i \begin{bmatrix} \mathbf{A_{uu}} & \mathbf{A_{uq}} \\ \kappa \mathbf{A_{uq}}^T & \mathbf{A_{qq}} \end{bmatrix}_i^{-1} \begin{bmatrix} \mathbf{A_{u\hat{u}}} \\ \kappa \mathbf{A_{q\hat{u}}} \end{bmatrix}_i + \left[\mathbf{A_{\hat{u}\hat{u}}}\right]_i + \left[\mathbf{A}_{\mathbf{\hat{u}\hat{u}}}^R\right]_i$$

and

$$\hat{\mathbf{f}} = \mathop{\mathrm{A}}_{i=1}^{n_{el}} \left[\mathbf{f_{\hat{u}}}\right]_i + \left[\mathbf{f}_{\mathbf{\hat{u}}}^R\right]_i - \begin{bmatrix} \mathbf{A}_{\mathbf{u\hat{u}}}^T & \mathbf{A}_{\mathbf{q\hat{u}}}^T \end{bmatrix}_i \begin{bmatrix} \mathbf{A_{uu}} & \mathbf{A_{uq}} \\ \kappa \mathbf{A_{uq}}^T & \mathbf{A_{qq}} \end{bmatrix}_i^{-1} \begin{bmatrix} \mathbf{f_u} \\ \kappa \mathbf{f_q} \end{bmatrix}_i$$

2 Solution

Given equation:

$$u(x,y) = \cosh\left(a\sin\left(\kappa\,x\right)^2 + b\,\cos\left(\pi\,\left(\gamma\,x^2 - y^3\right)\right)\right) \tag{2}$$

and following conditions: $\kappa = 4, \gamma = 2, a = -0.2$ and, $b = 0.8$

Plot for the equation (2), and take it as a reference.

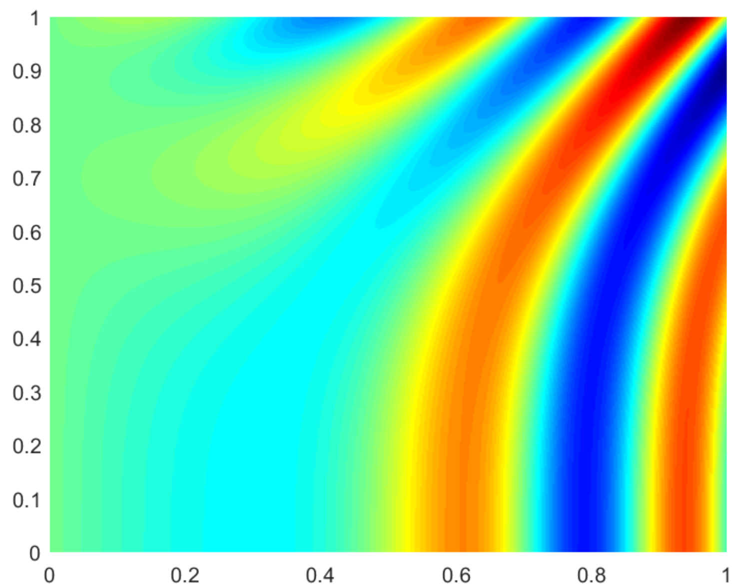


Figure 2.1: Plot for the equation

with the help of MATLAB analytic expression for u_D, t, and g were computed.

$$u_D = \begin{cases} \cosh\left(\frac{4\,\cos\left(\pi\,y^3\right)}{5}\right) & x = 0 \\ \cosh\left(\frac{4\,\cos\left(2\,\pi\,x^2\right)}{5} - \frac{\sin(4\,x)^2}{5}\right) & y = 0 \end{cases}$$

$$t = -4\sinh\left(\frac{4\,\cos\left(\pi\,\left(2\,\overline{x}^2 - \overline{y}^3\right)\right)}{5} - \frac{\sin\left(4\,\overline{x}\right)^2}{5}\right)\left(\frac{8\,\cos\left(4\,\overline{x}\right)\,\sin\left(4\,\overline{x}\right)}{5} + \frac{16\,\pi\,\sin\left(\pi\,\left(2\,\overline{x}^2 - \overline{y}^3\right)\right)\,\overline{x}}{5}\right)$$

$$g = 2\cosh\left(\frac{4\,\cos\left(\pi\,\left(2\,x^2 - y^3\right)\right)}{5} - \frac{\sin\left(4\,x\right)^2}{5}\right)$$

$$-\frac{48\,\pi}{5}\sinh\left(\frac{4\,\cos\left(2\,\pi\,\overline{x}^2 - \pi\,\overline{y}^3\right)}{5} - \frac{\sin\left(4\,\overline{x}\right)^2}{5}\right)\,\sin\left(2\,\pi\,\overline{x}^2 - \pi\,\overline{y}^3\right)\,\overline{y}^2$$

All these equations were introduced in the existing functions `analyticalPoisson.m` and `sourcePoisson.m` for the computations of the source term and the exact solution, which also includes the computations for the Dirichlet boundary values. Also two new functions `neumanPoisson.m` and `robinPoisson.m` were created to to accommodate t and g.

After that to apply boundry conditions, seperation of faces were performed with new function `ExtFace_class.m`. This new function divides the external boundary faces into Neumann, Dirichlet and Robin.
analytical solution obtained was,

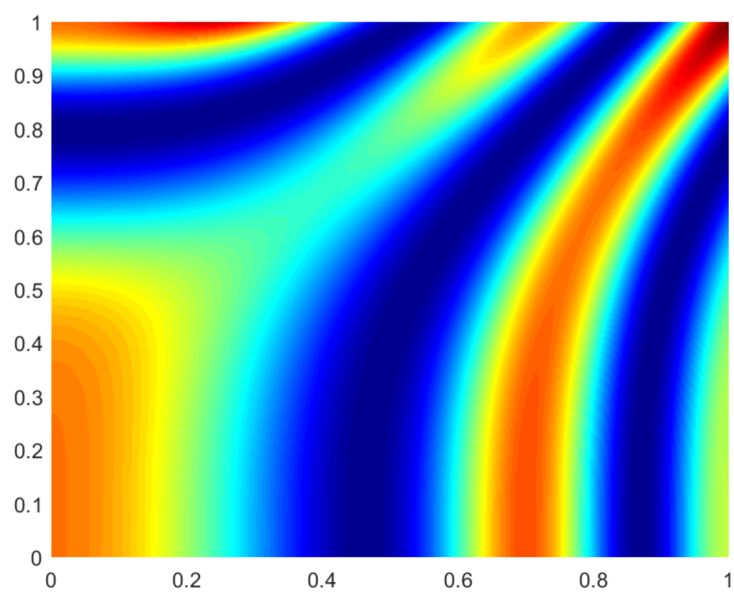
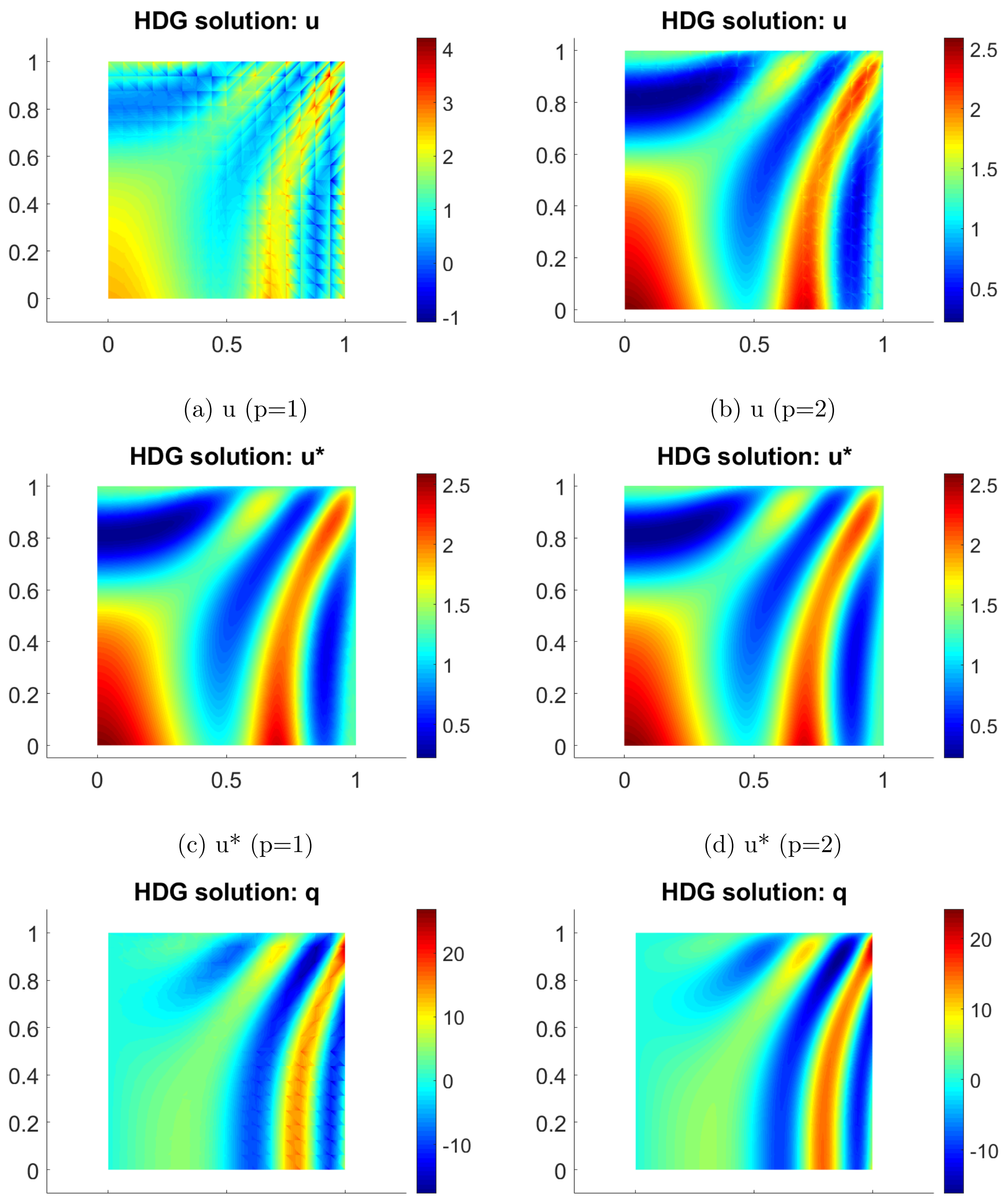


Figure 2.2: Analytical solution (u)

End results obtained, it is shown for a linear approximation different meshes:



(a) u (p=1) (b) u (p=2)

(c) u* (p=1) (d) u* (p=2)

(e) q (p=1) (f) q (p=2)

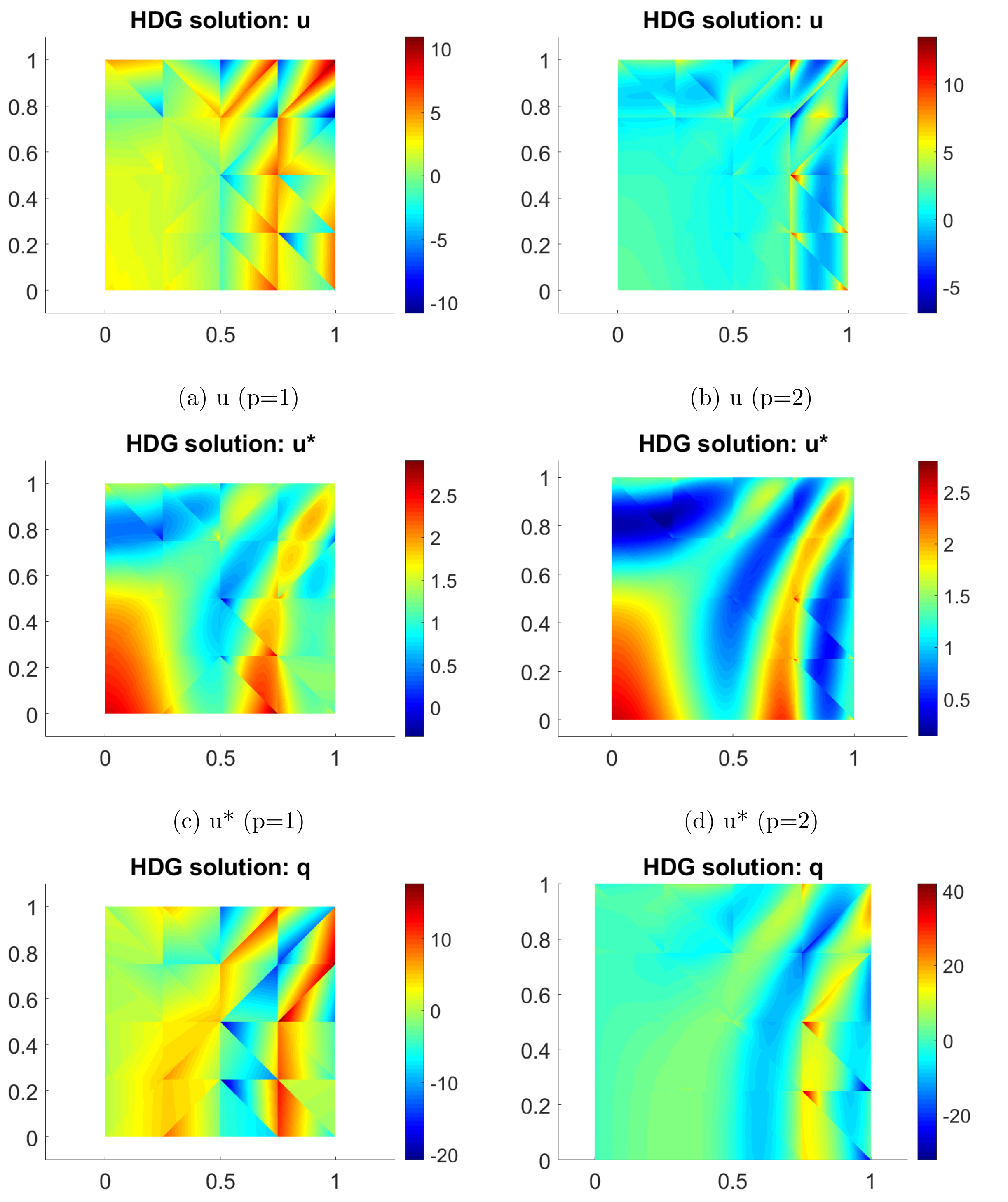
Figure 2.3: Comparison of the solution u, the postprocessed solution u*, and q using a linear and quadratic approximation for 1024 elements.

Following table shows error obtained for different cases

	1024 elements	
	p = 1	p = 2
Error u	5.221895e-01	4.551095e-01
Error u*	4.537080e-01	4.541246e-01
Error q	6.615271e+00	6.628339e+00

Table 1: Comparison of the error of the solution u, the postprocessed solution u*, and q using a linear and quadratic approximation for 1024 elements.

As seen from the error table with 1024 elements linear approximation performs better then quadratic. while for 64 elements quadratic approximation works better.



(a) u (p=1)
(b) u (p=2)
(c) u* (p=1)
(d) u* (p=2)
(e) q (p=1)
(f) q (p=2)

Figure 2.4: Comparison of the solution u, the postprocessed solution u*, and q using a linear and quadratic approximation for 64 elements.

Following table shows error obtained for different cases

	64 elements	
	p = 1	p = 2
Error u	1.940836e+00	1.205015e+00
Error u*	4.159823e-01	4.480802e-01
Error q	5.218238e+00	6.454382e+00

Table 2: Comparison of the error of the solution u, the postprocessed solution u*, and q using a linear and quadratic approximation for 64 elements.

Thus from above plots we can conclude that higher the number of elements we can reduce the degree of polynomial for optimal solution.

The convergence study was performed on u, u* and q for the value of κ =1 to 4 in the $\mathcal{L}_2(\Omega)$ norm.

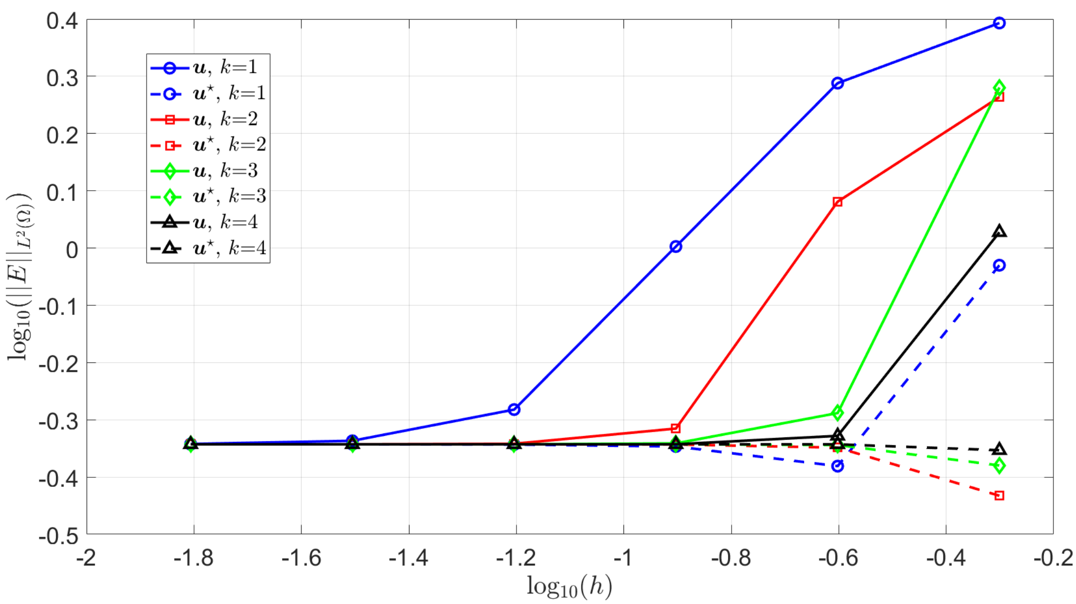


Figure 2.5: Error of the solution and the post-processed solution in the $\mathcal{L}_2(\Omega)$ norm as a function of the characteristic element size h for different values of the approximation degree p

Various colors represent the degree of polynomial. As can be seen as the number of elements in the mesh increases, the error decreases up to certain number but after that error changes sign and then increases.The improvements of the HDG methodology in the postprocess computation, where the solution u * that is shown with dashed lines gains precision can be seen converging and produces improved results.

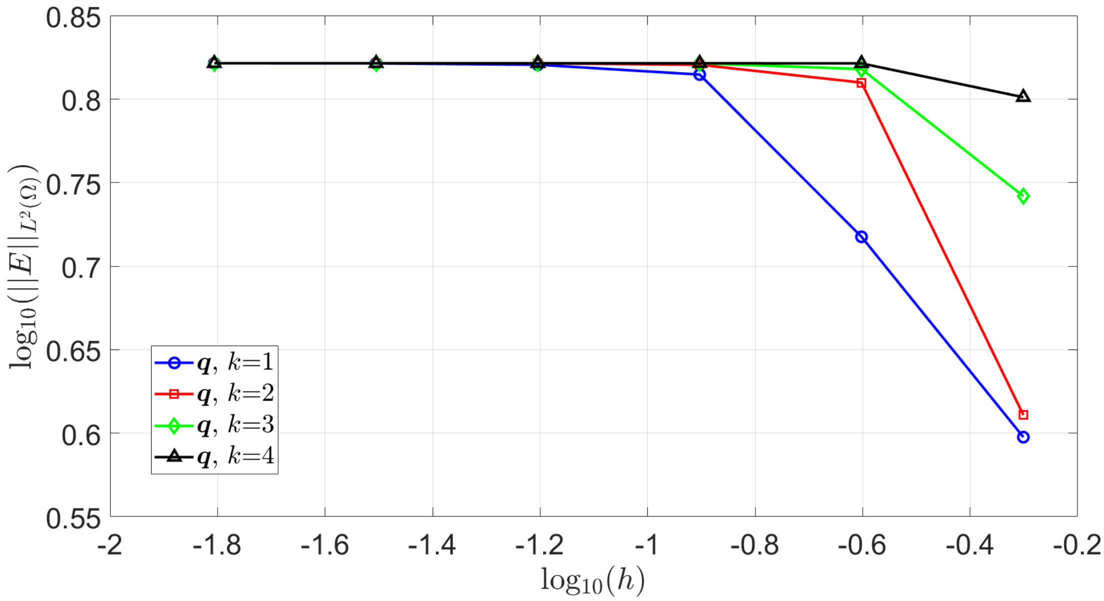


Figure 2.6: Error of q in the $\mathcal{L}_2(\Omega)$ norm as a function of the characteristic element size h for different values of the approximation degree p