Universitat Politècnica de Catalunya

Master of Science in Computational Mechanics

Finite Elements in Fluids

Assignment 7
Unsteady Incompressible Flow

Authors:
Carlos Eduardo Ribeiro Santa Cruz Mendoza

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UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

1 Unsteady Cavity Flow

The governing equations for a unsteady, isothermal and incompressible fluid flow are stated on Equation 1.1 and are regarded as the Navier-Stokes equations.

$$\begin{cases} \boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nabla p = \boldsymbol{b} & \text{in } \Omega \\ \nabla\cdot\boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v} = \boldsymbol{v}_D & \text{on } \Gamma_D \\ \boldsymbol{n}\cdot\boldsymbol{\sigma} = \boldsymbol{t} & \text{on } \Gamma_N \end{cases} \tag{1.1}$$

The first equation, namely the balance of momentum, is composed respectively of the unsteady term, the viscous term, the convective term, the pressure gradient and the external body forces.

The Navier Stokes equations can be applied to model a 2D flow inside a cavity, where the “lid” (at $y = 1$) has a velocity $v^* = 1$ while all the other walls have velocity zero. Thus, with the cavity completely filled by fluid and applying the no-slip condition, we can state the boundary conditions as depicted on Figure 1.1. The Reynolds number is assumed to be $Re = 100$ and the kinematic viscosity $\nu = 0,01$.

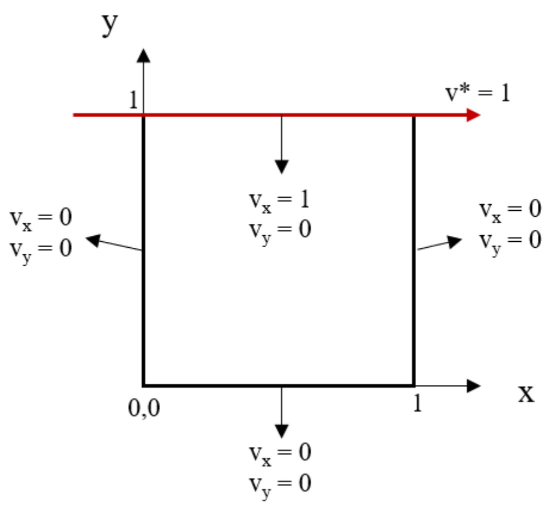


Figure 1.1: Domain and boundary conditions of the cavity problem

Given that we’re dealing only with Dirichlet boundary conditions, only relative pressure fields can be obtained. Thus, a reference value of zero was assigned to the origin.

The stated problem and corresponding governing equations are discretized in space via standard Galerkin formulation as shown on Equation 1.2.

$$\begin{cases} \boldsymbol{M}\dot{\boldsymbol{u}}(t) + [\boldsymbol{K} + \boldsymbol{C}(v)]\boldsymbol{u} + \boldsymbol{G}p = \boldsymbol{b} \\ \boldsymbol{G}^T\boldsymbol{u}(t) = 0 \\ \boldsymbol{u}(0) = \boldsymbol{v}(0) - \boldsymbol{v}_D(0) \end{cases} \tag{1.2}$$

where $\boldsymbol{M}$ is the mass matrix, defined after the Galerkin discretization of the unsteady term as shown in Equation 1.3.

$$\boldsymbol{v}_t = \boldsymbol{w} \cdot (\frac{\partial}{\partial t}\begin{bmatrix} v_x \\ v_y \end{bmatrix}) \Rightarrow \boldsymbol{M} = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ \vdots & \vdots \\ N_n & 0 \\ 0 & N_n \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & ... & N_n & 0 \\ 0 & N_1 & 0 & N_2 & ... & 0 & N_n \end{bmatrix} \tag{1.3}$$

The discretization in time can occur through several different methods, from which the *Semi-implicit* and the *Chorin-Temam* were chosen to be evaluated on the present report.

1.1 Semi-implicit discretization

The Semi-implicit method can be obtained starting from the *Theta method* time discretization given on Equation 1.4.

$$\frac{\Delta \boldsymbol{u}}{\Delta t} - \theta \Delta \boldsymbol{u}_t = \boldsymbol{u}_t^n \tag{1.4}$$

where $\Delta \boldsymbol{u} = \boldsymbol{u}^{n+1} - u^n$, n represents the time step and θ defines if the method will be explicit (note that with $\theta = 0$ the increment of the solution depends only on the previous time step) or implicit.

The method consists on finding the function increment ($\Delta \boldsymbol{u}$) by substituting the original partial differential equation on Equation 1.4. To do so, the first equation of the System 1.2 is rewritten as:

$$\boldsymbol{u}_t = \frac{\boldsymbol{b} - (\boldsymbol{K} + \boldsymbol{C}(v))\boldsymbol{u} - \boldsymbol{G}p}{\boldsymbol{M}} \tag{1.5}$$

Allowing for the substitution on Equation 1.4:

$$\frac{\Delta \boldsymbol{u}}{\Delta t} - \theta\Big(\frac{-(\boldsymbol{K} + \boldsymbol{C}^{n+1})\boldsymbol{u}^{n+1} + (\boldsymbol{K} + \boldsymbol{C}^n)\boldsymbol{u}^n - \boldsymbol{G}p^{n+1} + \boldsymbol{G}p^n}{\boldsymbol{M}}\Big) = \frac{\boldsymbol{b} - (\boldsymbol{K} + \boldsymbol{C}^n)\boldsymbol{u}^n - \boldsymbol{G}p^n}{\boldsymbol{M}} \tag{1.6}$$

The semi-implicit method consists of solving the system implicitly, but linearizing the convection term by treating it explicitly, that is, evaluating $\boldsymbol{C}^{n+1}$ as $\boldsymbol{C}^{n}$. Consequently, Equation 1.6 becomes:

$$\frac{\Delta \boldsymbol{u}}{\Delta t} + \theta\Big(\frac{(\boldsymbol{K}+\boldsymbol{C}^n)\Delta\boldsymbol{u} + \boldsymbol{G}\Delta p}{\boldsymbol{M}}\Big) = \frac{\boldsymbol{b} - (\boldsymbol{K}+\boldsymbol{C}^n)\boldsymbol{u}^n - \boldsymbol{G}p^n}{\boldsymbol{M}} \tag{1.7}$$

or, rearranging:

$$\big(\boldsymbol{M} + \theta\Delta t(\boldsymbol{K}+\boldsymbol{C}^n)\big)\Delta\boldsymbol{u} + \Delta t\theta\boldsymbol{G}\Delta p = \Delta t\big(\boldsymbol{b} - (\boldsymbol{K}+\boldsymbol{C}^n)\boldsymbol{u}^n - \boldsymbol{G}p^n\big) \tag{1.8}$$

Hence, since the incompressibility imposes divergence free flows at all time steps and given the additive property of the divergence we can state the modified system of equations:

$$\begin{cases} \big(\boldsymbol{M} + \theta\Delta t(\boldsymbol{K}+\boldsymbol{C}^n)\big)\Delta\boldsymbol{u} + \Delta t\theta\boldsymbol{G}\Delta p = \Delta t\big(\boldsymbol{b} - (\boldsymbol{K}+\boldsymbol{C}^n)\boldsymbol{u}^n - \boldsymbol{G}p^n\big) \\ \boldsymbol{G}^T\Delta\boldsymbol{u} = 0 \end{cases} \tag{1.9}$$

Which can be written in Matrix notation as:

$$\begin{bmatrix} \boldsymbol{M} + \theta\Delta t(\boldsymbol{K}+\boldsymbol{C}^n) & \Delta t\theta\boldsymbol{G} \\ \boldsymbol{G}^T & 0 \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{u} \\ \Delta\boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \Delta t\big(\boldsymbol{b} - (\boldsymbol{K}+\boldsymbol{C}^n)\boldsymbol{u}^n - \boldsymbol{G}p^n\big) \\ \boldsymbol{0} \end{bmatrix} \tag{1.10}$$

The linear system obtained on Equation 1.10 is, then, solved each iteration, providing the increment of the solution for a given time step.

1.2 Chorin-Temam discretization

The Chorin-Temam method is a projection method, calculating the velocity and pressure fields separately through a two-step computation. Equation 1.2 "split" in two parts: the first of which considers the viscous and convective terms (neglecting pressure and the incompressibility constraint) to calculate an intermediate velocity, employed on the next step to find the final pressure and velocity fields, now subject to incompressibility (but neglecting the viscous and convective terms, as well as the body forces).

The development of the first step is presented on Equation 1.11. The final equation is, then, solved to provide for $\boldsymbol{u}^*$.

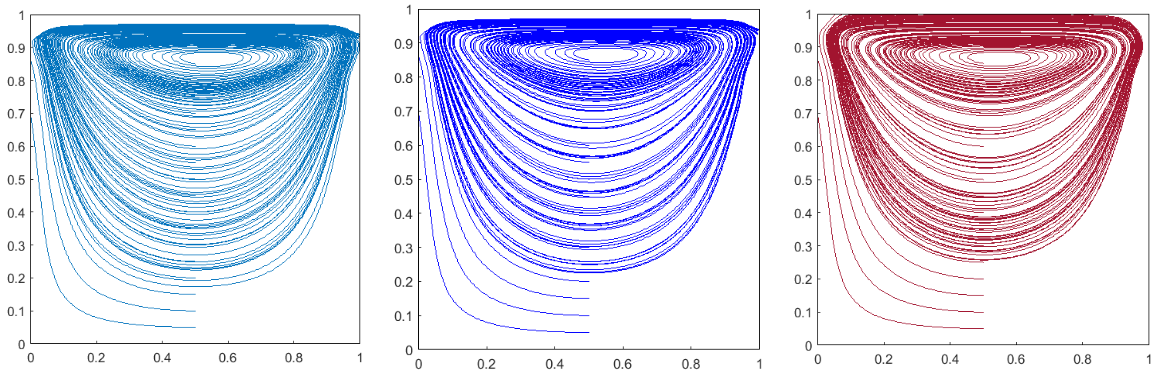
$$\begin{gathered} \boldsymbol{M}\boldsymbol{u}_t + (\boldsymbol{K}+\boldsymbol{C})\boldsymbol{u} = \boldsymbol{b} \\ \big(\boldsymbol{M} + \Delta t(\boldsymbol{K}+\boldsymbol{C}^n)\big)\boldsymbol{u}^* = \Delta t\boldsymbol{b} + \boldsymbol{M}\boldsymbol{u}^n \end{gathered} \tag{1.11}$$

The development of the second step is presented on Equation 1.12, where, finally, the obtained linear system is solved to compute the pressure and velocity fields corresponding to the current time step.

$$
\begin{gathered}
\boldsymbol{M}\boldsymbol{u}_t + \boldsymbol{G}p = 0 \\
\boldsymbol{M}\boldsymbol{u}^{n+1} + \Delta t\boldsymbol{G}p^{n+1} = \boldsymbol{M}\boldsymbol{u}^* \\
\begin{bmatrix} \boldsymbol{M} & \Delta t\boldsymbol{G} \\ \boldsymbol{G}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{n+1} \\ \boldsymbol{p}^{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}\boldsymbol{u}^* \\ 0 \end{bmatrix}
\end{gathered}
\tag{1.12}
$$

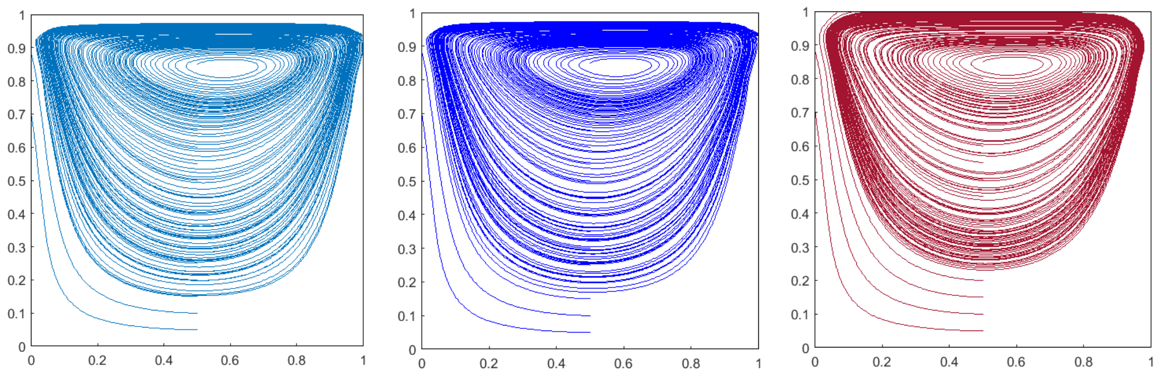
1.3 Results

The domain was discretized in a regular 10x10 mesh of Q2Q1 elements. Several time step sizes were tested and, realizing that it didn't affect the final result, a time step of $\Delta t = 0,01s$ was chosen to allow the development of the flow to be easily followed. The test cases are the semi-implicit method with $\theta = 0,5$ and $\theta = 1$ and the Chorin-Temam method. The results for the aformentioned cases are given from Figures 1.2 and 1.6.



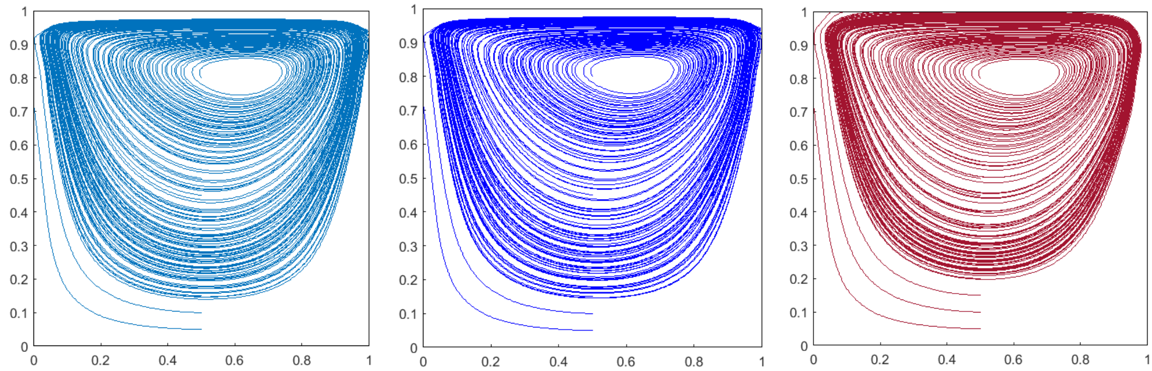
(a) $\theta = 0,5$ (b) $\theta = 1$ (c) Chorin-Temam

Figure 1.2: Streamlines for Time = 0,25 s



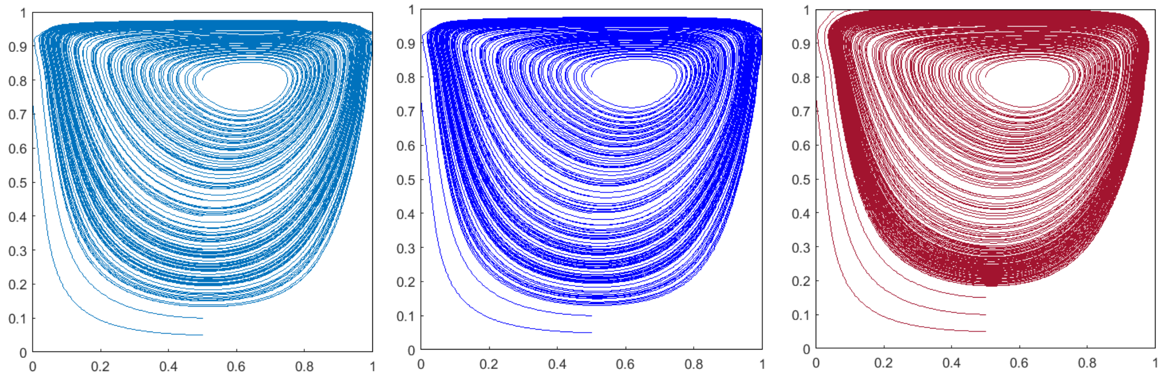
(a) $\theta = 0,5$ (b) $\theta = 1$ (c) Chorin-Temam

Figure 1.3: Streamlines for Time = 0,50 s



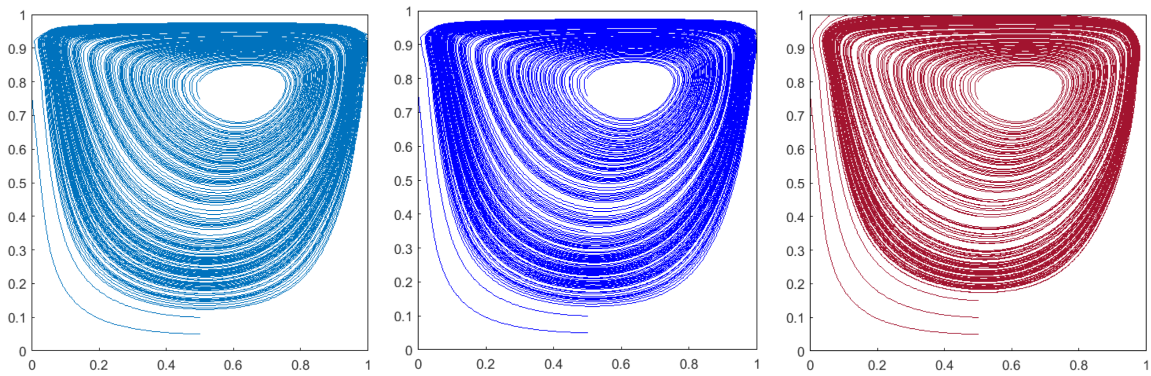
(a) $\theta = 0,5$ (b) $\theta = 1$ (c) Chorin-Temam

Figure 1.4: Streamlines for Time = 1,00 s



(a) $\theta = 0,5$ (b) $\theta = 1$ (c) Chorin-Temam

Figure 1.5: Streamlines for Time = 1,50 s

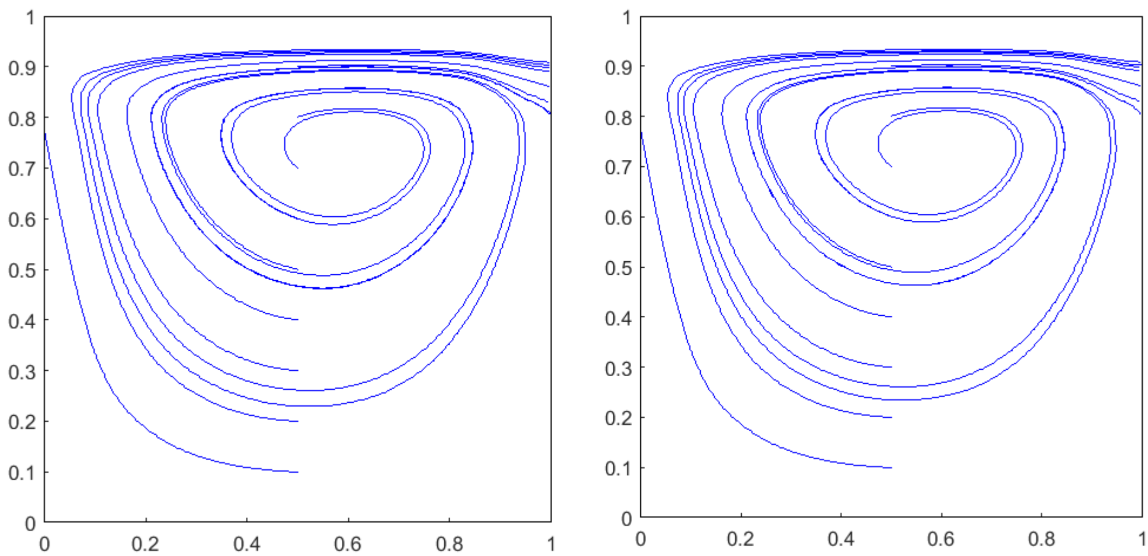


(a) $\theta = 0,5$ (b) $\theta = 1$ (c) Chorin-Temam

Figure 1.6: Streamlines for Time = 2,00 s

It's clear that the results for all methods are very similar in every time-step. The only difference worth noticing is that the semi-implicit method computed the flow to have reached a further depth of the cavity by $T = 2s$ ($y \approx 1,5$ against $y \approx 2$ on the Chorin-Temam). This could be attributed to the different handling of the viscous and convective terms on both methods.

The semi-implicit method for $\theta < 0,5$ only converged with the usage of the SUPG stabilization method (Q1Q1 elements). But as we can see on Figure 1.7, the results are virtually the same when comparing $\theta = 0$ and $\theta = 1$.



(a) $\theta = 0$ (b) $\theta = 1$

Figure 1.7: Streamlines for Time = 2,00 s