Finite Elements in Fluid

Homework 7: Hybridizable Discontinuous Galerkin

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Consider the domain $\Omega = [0,1]^2$ such that $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$ with $\Gamma_D \cap \Gamma_N$=0, $\Gamma_D \cap \Gamma_R$=0 and $\Gamma_N \cap \Gamma_R$=0. More precisely, set

$$\Gamma_N := \{(x,y) \in \mathbb{R}^2 \colon\ y = 0\},$$

$$\Gamma_R := \{(x,y) \in \mathbb{R}^2 \colon\ x = 1\},$$

$$\Gamma_D := \ \partial\Omega \backslash (\Gamma_N \cup \Gamma_R).$$

The following second-order linear scalar partial differential equation is defined.

$$\begin{cases} -\nabla\cdot(k\nabla u) = s & in\Omega \\ u = u_D & on\ \Gamma_D \\ n\cdot(k\nabla u) = t & on\ \Gamma_N \\ n\cdot(k\nabla u) + \gamma u = g & on\ \Gamma_R \end{cases} \qquad (P)$$

Where k and γ are the diffusion and convection coefficients, respectively, n is the outward unit normal vector to the boundary, s is a volumetric source term and u_D , t and g are the Dirichlet, Neumann and Robin data imposed on the corresponding portions of the boundary $\partial\Omega$.

1. Write the HDG formulation of the problem (P). More precisely, derive the HDG strong and weak forms of the local and global problem.

   We define two equivalent problem:

   1st, Element-by-element problem (Local-Dirichlet).

$$\begin{cases} \nabla\cdot q_i = s & in\Omega_i \\ q_i + k\nabla q_i = 0 & in\Omega_i \\ u_i = u_D & in\ \partial\Omega_i \cap \Gamma_D \\ u_i = \hat{u} & on\ \partial\Omega_i \setminus \Gamma_D \end{cases}$$

   2nd, Global problem to determine $\hat{u}$ (Neumann-Robin Transmission condition)

$$\begin{cases} [\![n\cdot q]\!] = 0 & on\ \Gamma \\ n\cdot q = -t & on\ \Gamma_N \\ -q\cdot n + \gamma u = g & on\ \Gamma_R \end{cases}$$

   For problem 1st, we think f as s in the following,

$$-(\nabla v, q_i)_{\Omega_i} + < \mathrm{v}, n_i\cdot\hat{q}_i >_{\partial\Omega_i} = (v, \mathrm{f})_{\Omega_i}$$

$$-(w, q_i)_{\Omega_i} + (\nabla\cdot w, u_i)_{\Omega_i} = < u_i\cdot w, u_D >_{\partial\Omega_i\cap\Gamma_D} + < n_i\cdot w, \hat{u} >_{\partial\Omega_i\backslash\Gamma_D}$$

$$n_i\cdot\hat{q}_i = \begin{cases} n_i\cdot q_i + \tau_i(u_i - u_D) & on\ \partial\Omega_i\setminus\Gamma_D \\ n_i\cdot q_i + \tau_i(u_i - \hat{u}) & elsewhere \end{cases} \qquad (1)$$

$$-(\nabla v, q_i)_{\Omega_i} + < \mathrm{v}, \tau_i u_i >_{\partial\Omega_i} + < \mathrm{v}, n_i \cdot \hat{q}_i >_{\partial\Omega_i}$$
$$= (v, \mathrm{f})_{\Omega_i} + < v, \tau_i u_D >_{\partial\Omega_i \cap \Gamma_D} + < v, \tau_i \hat{u} >_{\partial\Omega_i \setminus \Gamma_D}$$

$$-(w, q_i)_{\Omega_i} + (\nabla\cdot w, u_i)_{\Omega_i} = < u_i \cdot w, u_D >_{\partial\Omega_i \cap \Gamma_D} + < n_i \cdot w, \hat{u} >_{\partial\Omega_i \setminus \Gamma_D}$$

For problem 2nd,

$$\sum_{i=1}^{n_{el}} < \mu, n_i \cdot \hat{q}_i >_{\partial\Omega_i \setminus \partial\Omega} + \sum_{i=1}^{n_{el}} < \mu, n_i \cdot \hat{q}_i + \mathrm{t} >_{\partial\Omega_i \cap \Gamma_D}$$
$$+ \sum_{i=1}^{n_{el}} < \mu, n_i \cdot \hat{q}_i - \gamma u + g >_{\partial\Omega_i \cap \Gamma_R} = 0$$

And then from (1), we get

$$\sum_{i=1}^{n_{el}} < \mu, \tau_i u_i^h >_{\partial\Omega_i \setminus \Gamma_D} + \sum_{i=1}^{n_{el}} < \mu, n_i \cdot q_i^h >_{\partial\Omega_i \setminus \Gamma_D} - \sum_{i=1}^{n_{el}} < \mu, \tau_i \hat{u}^h >_{\partial\Omega_i \setminus \Gamma_D}$$
$$- \sum_{i=1}^{n_{el}} < \mu, \gamma \hat{u}^h >_{\partial\Omega_i \cap \Gamma_R}$$
$$= \sum_{i=1}^{n_{el}} < \mu, -t >_{\partial\Omega_i \cap \Gamma_N} + \sum_{i=1}^{n_{el}} < \mu, -g >_{\partial\Omega_i \cap \Gamma_R}$$

Now, integrating by parts the first term of the LHS of equation for the global problem and leaving the element values of the flux q on the boundary into the interior,

$$-(v, \nabla\cdot q_i)_{\Omega_i} + < \mathrm{v}, \tau_i u_i >_{\partial\Omega_i} = (v, \mathrm{f})_{\Omega_i} + < v, \tau_i u_D >_{\partial\Omega_i \cap \Gamma_D} + < v, \tau_i \hat{u} >_{\partial\Omega_i \setminus \Gamma_D}$$

$$-(w, q_i)_{\Omega_i} + (\nabla\cdot w, u_i)_{\Omega_i} = < n_i \cdot w, u_D >_{\partial\Omega_i \cap \Gamma_D} + < n_i \cdot w, \hat{u} >_{\partial\Omega_i \setminus \Gamma_D}$$

Then, take Galerkin FE approximation,

$$-\left(v, \nabla\cdot q_i^h\right)_{\Omega_i} + < \mathrm{v}, \tau_i u_i^h >_{\partial\Omega_i}$$
$$= (v, \mathrm{f})_{\Omega_i} + < v, \tau_i u_D >_{\partial\Omega_i \cap \Gamma_D} + < v, \tau_i \hat{u}^h >_{\partial\Omega_i \setminus \Gamma_D}$$

$$-\left(w, q_i^h\right)_{\Omega_i} + \left(\nabla\cdot w, u_i^h\right)_{\Omega_i} = < n_i \cdot w, u_D >_{\partial\Omega_i \cap \Gamma_D} + < n_i \cdot w, \hat{u}^h >_{\partial\Omega_i \setminus \Gamma_D}$$

$$\sum_{i=1}^{n_{el}} < \mu, \tau_i u_i^h >_{\partial\Omega_i \setminus \Gamma_D} + \sum_{i=1}^{n_{el}} < \mu, n_i \cdot q_i^h >_{\partial\Omega_i \setminus \Gamma_D} - \sum_{i=1}^{n_{el}} < \mu, \tau_i \hat{u}^h >_{\partial\Omega_i \setminus \Gamma_D}$$
$$- \sum_{i=1}^{n_{el}} < \mu, \gamma \hat{u}^h >_{\partial\Omega_i \cap \Gamma_R}$$
$$= \sum_{i=1}^{n_{el}} < \mu, -t >_{\partial\Omega_i \cap \Gamma_N} + \sum_{i=1}^{n_{el}} < \mu, -g >_{\partial\Omega_i \cap \Gamma_R}$$

After the discretization of the global and local problem, it is obtained the following system of equation,

$$\begin{bmatrix} A_{uu} & A_{uq} \\ A_{uq}^T & A_{qq} \end{bmatrix} \begin{Bmatrix} u_i \\ q_i \end{Bmatrix} = \begin{Bmatrix} f_u \\ f_q \end{Bmatrix}_i + \begin{bmatrix} A_{u\hat{u}} \\ A_{q\hat{u}} \end{bmatrix} \hat{u}$$

$$\sum_i^{n_{el}} \left\{ \begin{bmatrix} A_{u\hat{u}}^T & A_{q\hat{u}}^T \end{bmatrix} \begin{Bmatrix} u_i \\ q_i \end{Bmatrix} + [A_{\hat{u}\hat{u}}]\hat{u}_i \right\} = \sum_i^{n_{el}} [f_{\hat{u}}]_i$$

Then replacing the first system of equation into the second equation, we get

$$\widehat{K} = \prod_i^{n_{el}} \begin{bmatrix} A_{u\hat{u}}^T & A_{q\hat{u}}^T \end{bmatrix}_i \begin{bmatrix} A_{uu} & A_{uq} \\ A_{uq}^T & A_{qq} \end{bmatrix}_i^{-1} \begin{bmatrix} A_{u\hat{u}} \\ A_{q\hat{u}} \end{bmatrix} + [A_{\hat{u}\hat{u}}]_i$$

$$\hat{f} = \prod_i^{n_{el}} [f_{\hat{u}}]_i - \begin{bmatrix} A_{u\hat{u}}^T & A_{q\hat{u}}^T \end{bmatrix}_i \begin{bmatrix} A_{uu} & A_{uq} \\ A_{uq}^T & A_{qq} \end{bmatrix}_i^{-1} \begin{Bmatrix} f_u \\ f_q \end{Bmatrix}_i$$

$$\widehat{K}\hat{u} = \hat{f}$$

The term $A_{\hat{u}\hat{u}}$ involves Robin conditions whereas the term $f_{\hat{u}}$ computes both Neumann and Robin boundary conditions.

2. Implement in the Matlab code provided in class the corresponding HDG solver.
    1. The degree of freedom of the unknows should be calculated according to the global face ID of the Dirichlet boundaries.
    2. To gather the types of faces, we have added an extra cell of extFaces. There are differences among Dirichlet, Robin and Neumann boundary conditions.
    3. In hdgMatrixpoission, the matrix of the global system needs to be computed as adding the extra terms to contain Neumann and Robin.
    4. In the postprocess, we should add the viscosity term.

3. Set $k = 8$ and $\gamma = 5$. Consider $u(x, y) = \exp[\sin(axy) - bcos(\ \gamma x + k\pi y)]$, with $a = 0.1$ and $b = 0.6$. Determine the analytical expressions of the data u_D , t and g in problem (P).

$$q = -\mathrm{k} \begin{bmatrix} \frac{du}{dx} & \frac{du}{dy} \end{bmatrix}$$

$$t = -\mathrm{k}\frac{du}{dx}$$

$$g = k\frac{du}{dx} + \gamma u$$

$$s = -k\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2}\right)$$

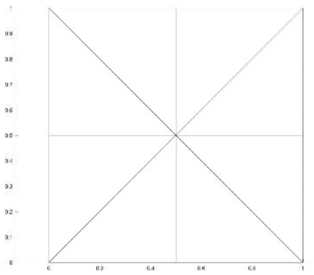
Take the given $u(x, y)$, yield

$$t = -k\ \cdot \exp[sin(axy) - bcos(\ \gamma x + k\pi y)] \cdot [cos(axy) \cdot ay + bsin(\gamma x + k\pi y) \cdot \gamma]$$

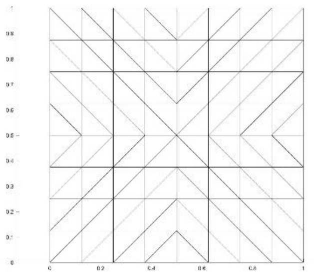
$$g = k\ \cdot \exp[sin(axy) - bcos(\ \gamma x + k\pi y)] \cdot [cos(axy) \cdot ay + bsin(\gamma x + k\pi y) \cdot \gamma] + \ \gamma\ \cdot \exp[\sin(axy) - bcos(\ \gamma x + k\pi y)]$$

$$s = -k\{\exp[sin(axy) - bcos(\gamma x + k\pi y)] \cdot [\cos(axy) \cdot ay + bsin(\gamma x + k\pi y) \cdot \gamma]^2 + \exp[sin(axy) - bcos(\gamma x + k\pi y)]\,[-\sin(axy)\,a^2 y^2 + bcos(\gamma x + k\pi y)\gamma^2] + \exp[sin(axy) - bcos(\gamma x + k\pi y)] \cdot [\cos(axy) \cdot ax + bsin(\gamma x + k\pi y) \cdot k\pi]^2 + \exp[sin(axy) - bcos(\gamma x + k\pi y)]\,[-\sin(axy)\,a^2 x^2 + bcos(\gamma x + k\pi y)k^2\pi^2]\}$$

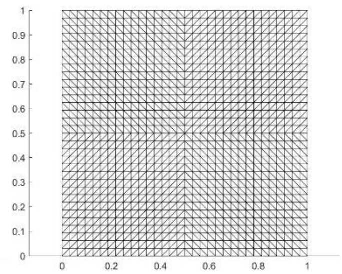
4. Solve problem (P) using HDG with different meshes and polynomial degrees of approximation. Starting from the plots provided by the Matlab codes, discuss the accuracy of the obtained solution u and of the postprocessed one u^*.



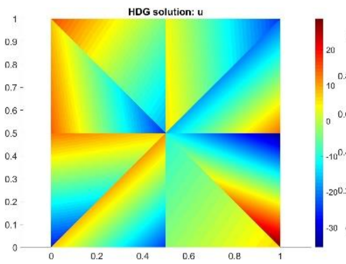
Mesh1



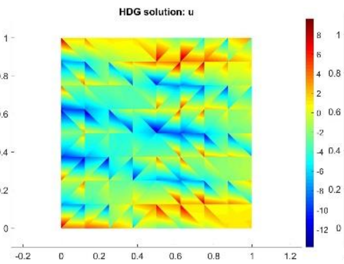
Mesh3



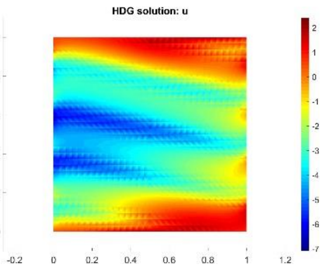
Mesh5



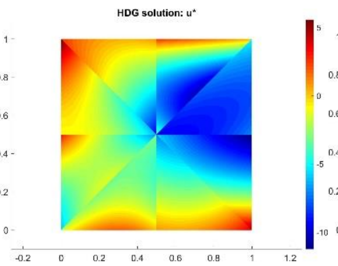
Mesh1 Degree1 u



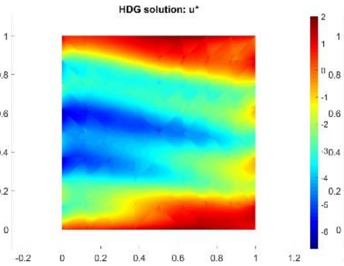
Mesh3 Degree1 u



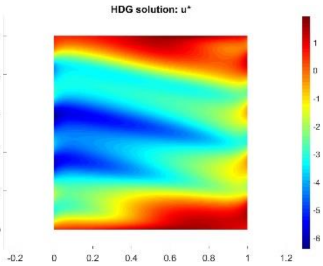
Mesh5 Degree1 u



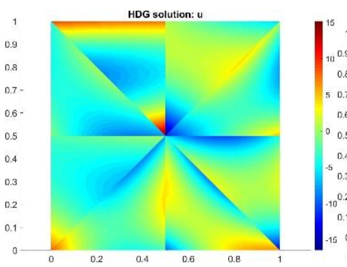
Mesh1 Degree1 u*



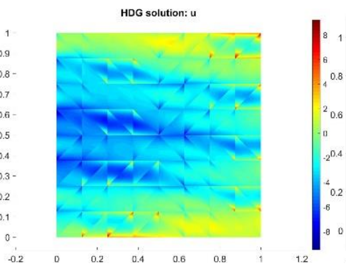
Mesh3 Degree1 u*



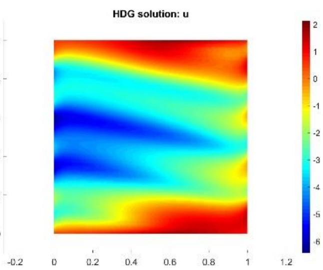
Mesh5 Degree1 u*



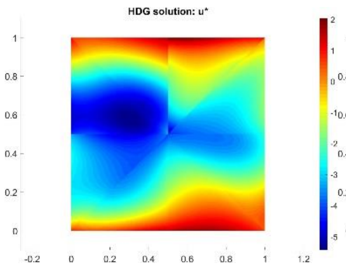
Mesh1 Degree2 u



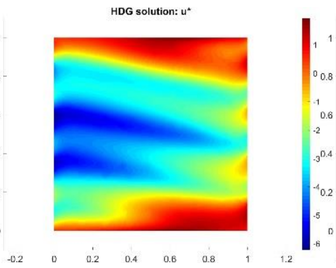
Mesh3 Degree2 u



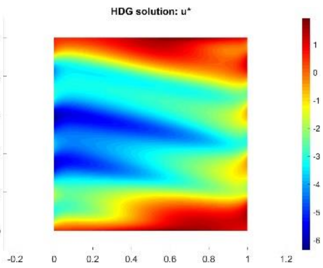
Mesh5 Degree2 u



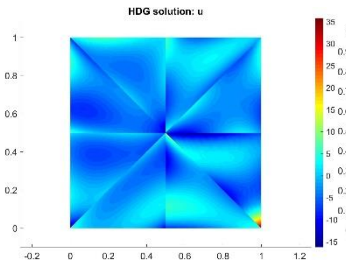
Mesh1 Degree2 u*



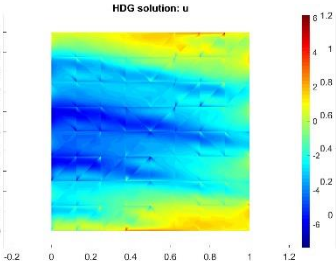
Mesh3 Degree2 u*



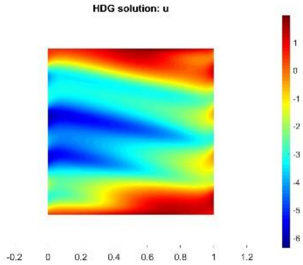
Mesh5 Degree2 u*



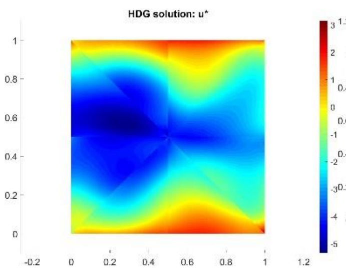
Mesh1 Degree3 u



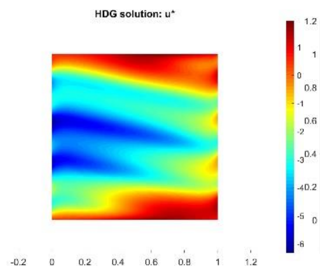
Mesh3 Degree3 u



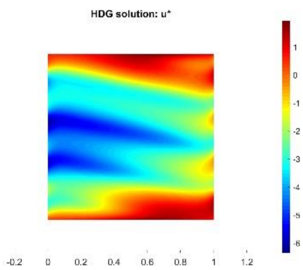
Mesh5 Degree3 u



Mesh1 Degree3 u*



Mesh3 Degree3 u*



Mesh5 Degree3 u*

mesh	degree	elements	$\|u-u^h\|_{L_2\Omega}$	$\|u-u*^h\|_{L_2\Omega}$
1	1	8	10.21733	5.76684
1	2	8	4.650495	3.684738
1	3	8	4.714951	3.76076
3	1	128	4.514119	3.870792
3	2	128	3.980082	3.868367
3	3	128	3.882971	3.86925
5	1	2048	3.880898	3.869294
5	2	2048	3.869497	3.869371
5	3	2048	3.869372	3.869371

Table 4.1 Comparing L_2 norm for HDG u and u* on different mesh and different degree

It is obviously that if the degree of the approximation increasing or the mesh elements increasing the error of the variables will decrease.

On the other hand, the accuracy of the postprocessed solution u* is higher than the one of HDG solution u in the same degree of the approximation and same mesh condition.

5. Compute the errors for u, q and u^* in L_2 -norm defined on the domain Ω. Perform a convergence study for the primal, u, mixed, q and postprocessed, u^* variables for a polynomial degree of approximation $k = 1, \dots, 4$. Discuss the obtained numerical results, starting from the theoretical results on the optimal convergence rates of HDG.

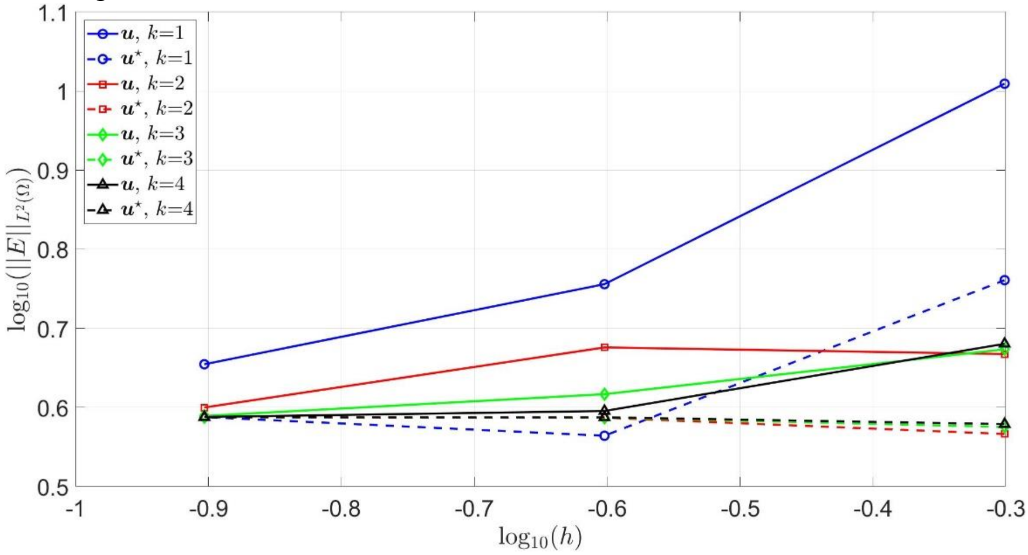


Figure 5.1 Convergence analysis for u and u* according k=1,2,3,4 in Mesh3

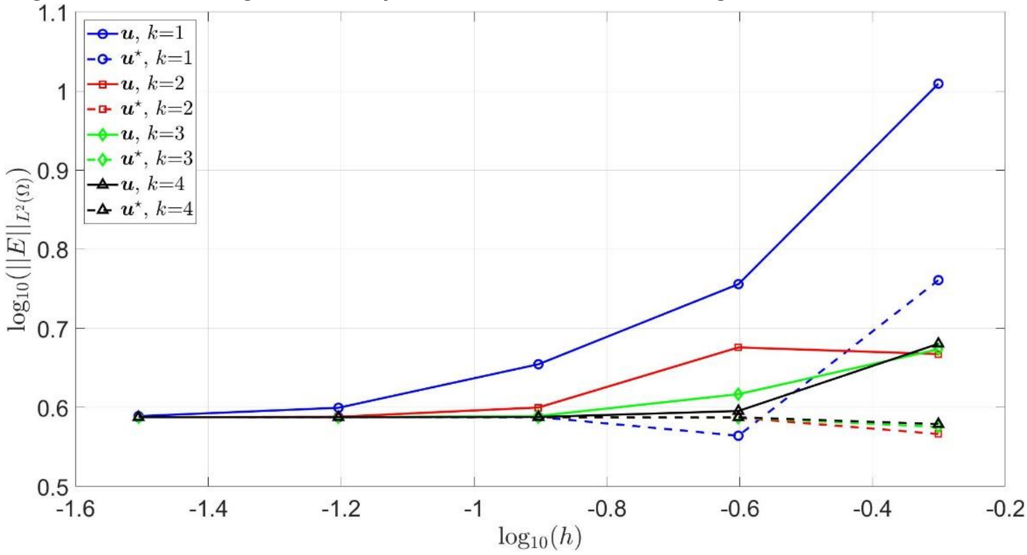


Figure 5.2 Convergence analysis for u and u* according k=1,2,3,4 in Mesh5

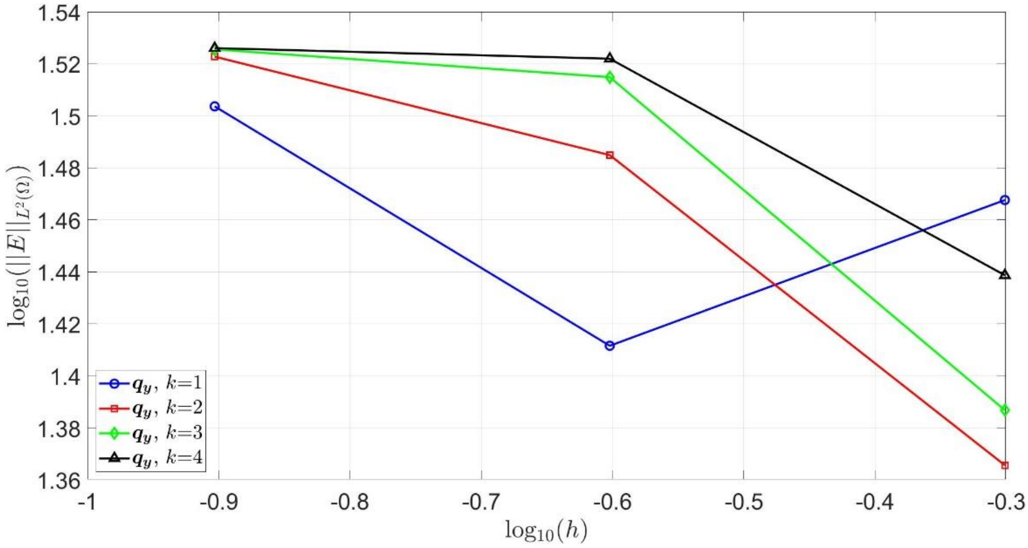


Figure 5.3 Convergence analysis for component x of q according k=1,2,3,4 in Mesh3

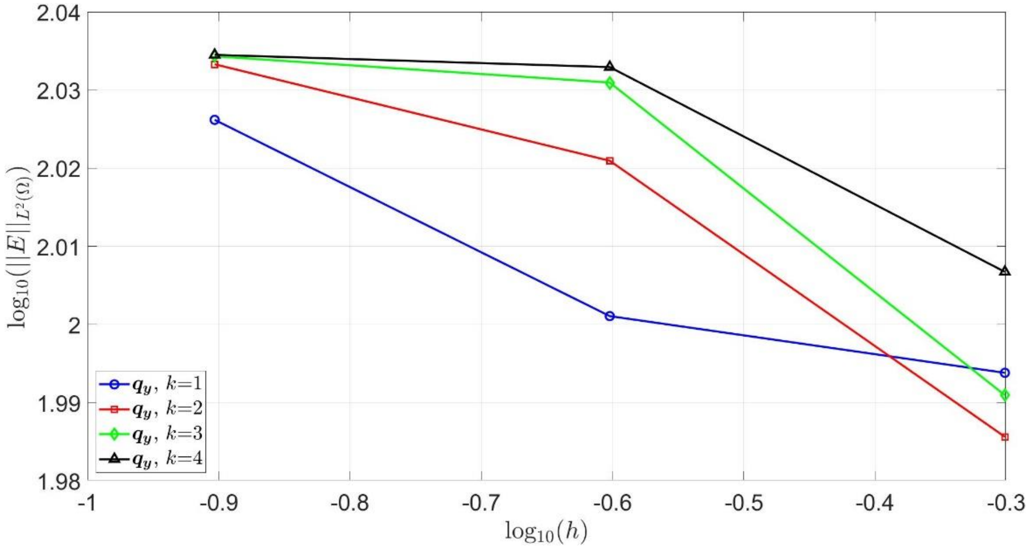


Figure 5.4 Convergence analysis for component y of q according k=1,2,3,4 in Mesh3

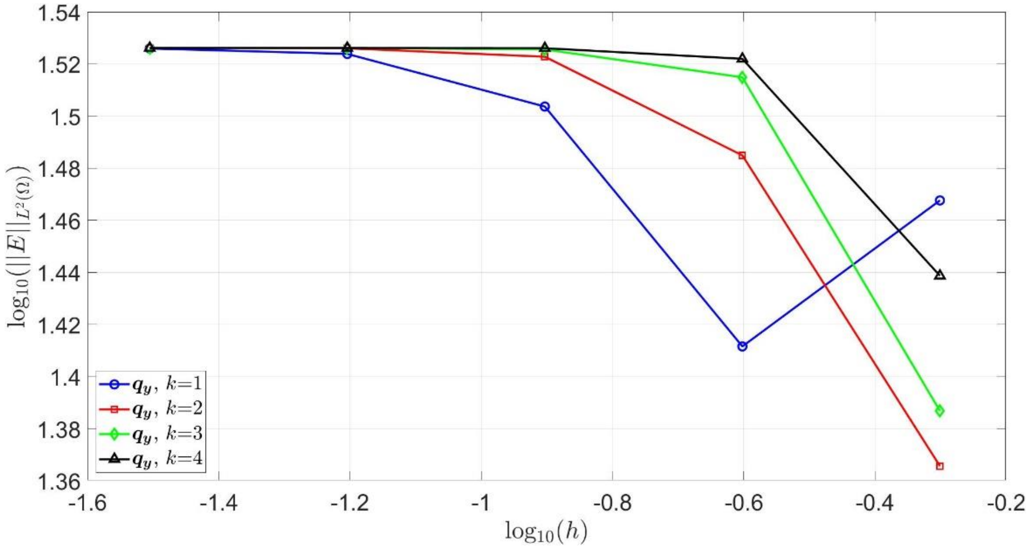


Figure 5.5 Convergence analysis for component x of q according k=1,2,3,4 in Mesh5

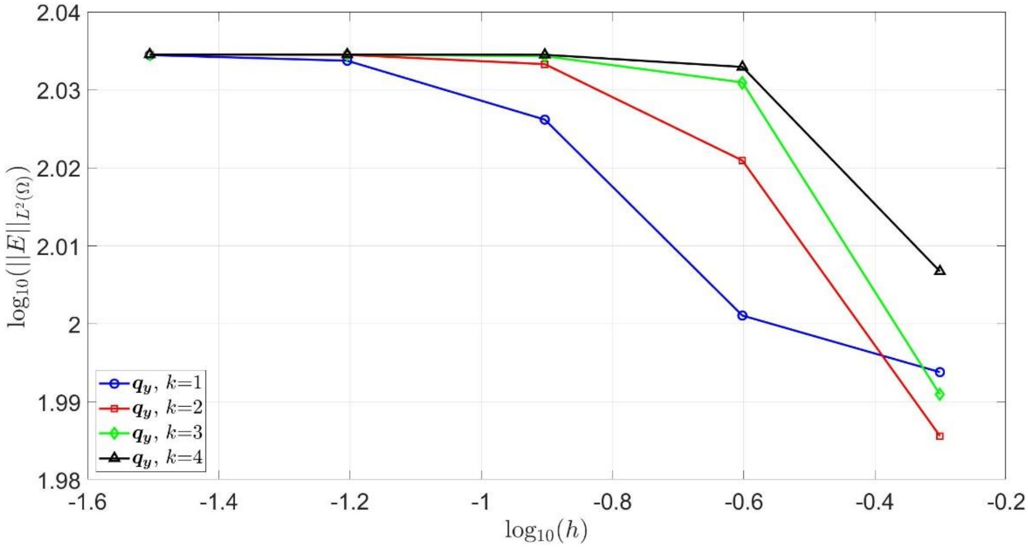


Figure 5.6 Convergence analysis for component y of q according k=1,2,3,4 in Mesh5

The above figures show the convergence of the L_2-norm error of u, q and u* for the polynomial degree of approximation from 1 to 4. T he convergence rate optimal (k+1) is for u and q while the super-convergence rate (k+2) is for postprocessed solution u*.

REFERENCE

[1] Lecture slides in Finite elements in fluid.
[2] Finite Element Methods for Flow Problems, Jean Donea and Antonio Huerta.
[3] Ruben Sevilla,' Hybridisable discontinuous Galerkin for second-order elliptic problems', Zienkiewicz Centre for Computational Engineering, College of Engineering, Swansea University, Bay Campus, SA1 8EN, Wales, UK,DG Summer School - Barcelona - July 2017