**Finite Elements in Fluid**

**Homework 6: Unsteady Navier-Stokes numerical examples**

**Ye Mao**

**Ye Mao, mao.ye@estudiant.upc.edu**
Master of Numerical methods on engineering - Universitat Politècnica de Catalunya

1. INTRODUCTION

Stokes equations

$$v_t - \nu\,\nabla^2 v + (v \bullet \nabla)v + \nabla p = b \qquad in\Omega \qquad (1)$$

$$\nabla \bullet v = 0 \qquad in\Omega \qquad (2)$$

$$v = v_D \qquad on\Gamma_D \qquad (3)$$

$$n \bullet \boldsymbol{\sigma} = t \qquad on\Gamma_D \qquad (4)$$

The first equation is balance of momentum equation. It is composed of the unsteady term, the viscous term, the convective term, the pressure gradient and external body forces.

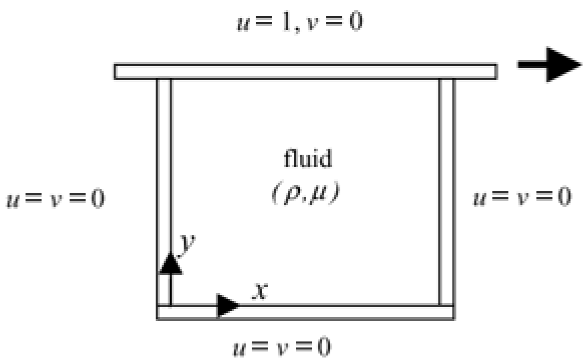


**Figure 1. Classical problem of the lid-driven square cavity flow.**

The Navier Stokes equations can be applied as a classical 2D problem which is the 2D flow inside of a cavity. According the above figure, while $y = 1$, the velocity in x direction is 1 and all the other walls' velocity are zero. We can assume the $R_e = 200$ $and\ the\ kinematic\ viscosity\ \nu = 0.005$, where the domain is $x, y \in [0,1]$.

Here, we only deal with Dirichlet boundary conditions and only relative pressure filed can be obtained. So, the we can set the reference value as zeros to the initial condition.

This stated problem and corresponding governing equations are discretized in space by standard Galerkin formulation as shown on the following equation.

$$\begin{cases} M\dot{u}(t) + \big(K + C(v)\big)u + Gp = b \\ G^T u(t) = 0 \\ u(0) = v(0) - v_D(0) \end{cases} \qquad (5)$$

Where $M$ is the mass matrix, it is defined as the Galerkin discretization of the unsteady term as following:

$$v_t = w \cdot \left(\frac{\partial}{\partial_t}\begin{bmatrix} v_x \\ v_y \end{bmatrix}\right) \tag{6}$$

$$M = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ \vdots & \vdots \\ N_n & 0 \\ 0 & N_n \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_n \end{bmatrix} \tag{7}$$

In this report, we choose the Semi-implicit and the Chorin-Temam methods which are the discretization in time.

2. OBJECTIVE

2.1 Complete the Navier-Stokes problem with the Semi-implicit and the Chorin-Temam methods which are the discretization in time.

3. METHODOLOGY AND RESULTS

3.1 Semi-implicit discretization

The first order Semi-implicit method can be obtained from the $\theta$ method time discretization as follow:

$$\frac{\Delta u}{\Delta t} - \theta \Delta u_t = u_t^n \tag{8}$$

Where $\Delta u = u^{n+1} - u^n$, $n$ is time step, $\theta = 0$, explicit FTCS; $\theta$=0.5 Crank-Nicolson; $\theta$=1 implicit BTCS.

From Equation 8, we get

$$\dot{u}(t) = \frac{b - \big(K + C(v)\big)u - Gp}{M} \tag{9}$$

Then substitute on Equation 8

$$\frac{\Delta u}{\Delta t} - \theta\left(\frac{-(K + C^{n+1})u^{n+1} + (K + C^n)u^n - Gp^{n+1} + Gp^n}{M}\right) = \frac{b - (K + C^n)u^n - Gp^n}{M} \tag{10}$$

The semi-implicit methods solve the implicitly system. In this case, we can linearize the convection term by explicitly one. That is evaluating $C^{n+1}$ as $C^n$. Then the Equation 10 becomes:

$$\frac{\Delta u}{\Delta t} - \theta\left(\frac{(K + C^n)\Delta u + Gp^n}{M}\right) = \frac{b - (K + C^n)u^n - Gp^n}{M} \tag{11}$$

Rearranging:

$$\big(M + \theta \Delta t(K + C^n)\big)\Delta u + \theta \Delta t G \Delta p = \Delta t(b - (K + C^n)u^n - Gp^n) \tag{12}$$

Due to the incompressibility affect the divergence free flows at all time steps and given the additive property of the divergence, we get:

$$\begin{cases}\left(M+\theta\Delta t(K+C^n)\right)\Delta u+\theta\Delta tG\Delta p=\Delta t(b-(K+C^n)u^n-Gp^n)\\ G^T\Delta u=0\end{cases} \quad (13)$$

Which can be written in Matrix notation as following:

$$\begin{bmatrix}M+\theta\Delta t(K+C^n) & \theta\Delta tG\\ G^T & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta p\end{bmatrix}=\begin{bmatrix}\Delta t(b-(K+C^n)u^n-Gp^n)\\ 0\end{bmatrix} \quad (14)$$

Now, the Equation 14 is the linear system. We can take iteration providing the increment of the solution for time step.

### 3.2 Chorin-Temam discretization

The Chorin-Temam projection method calculate the velocity and pressure fields separately through a two-step computation. Equation 5 splits into two parts: we consider the viscous and convective terms to calculate an intermediate velocity in the first. In this case we neglect pressure and incompressibility constraint. Then take them to the next step to find the final pressure and velocity fields. Now, we consider the incompressibility but neglecting the viscous and convective terms and body forces.

The development of 1st step is presented on Equation 15. Then the final equation is solved to provide for $u^*$

$$Mu_t+(K+C)u=b$$
$$\left(M+\Delta t(K+C^n)\right)u^*=\Delta tb+Mu^n \quad (15)$$

Then the development of second step is presented on Equation 16. Finally, we obtain linear system which is solved to compute the pressure and velocity corresponding to the current time step.

$$Mu_t+Gp=0$$
$$Mu^{n+1}+\Delta tGp^{n+1}=Mu^*$$
$$\begin{bmatrix}M & \Delta tG\\ G^T & 0\end{bmatrix}\begin{bmatrix}u^{n+1}\\ p^{n+1}\end{bmatrix}=\begin{bmatrix}Mu^*\\ 0\end{bmatrix} \quad (16)$$

### 3.3 Results

This domain is discretized in a regular mesh which is 10x10 by Q2Q1 elements. In this report, we test time step size as 0.1s 0.2s 0.5s 1s. Find that it does not affect the final result. Time step $\Delta t=0.01s$ is chosen, where using the test methods as semi-implicit, $\theta=0.5$ and $\theta=1$, and the Chorin-Temam one. All the results for the description above are given from the following Figures 2 to 5.

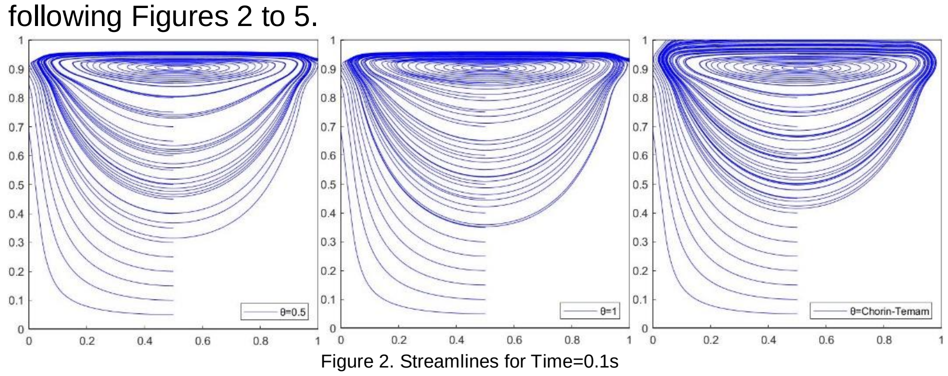

Figure 2. Streamlines for Time=0.1s

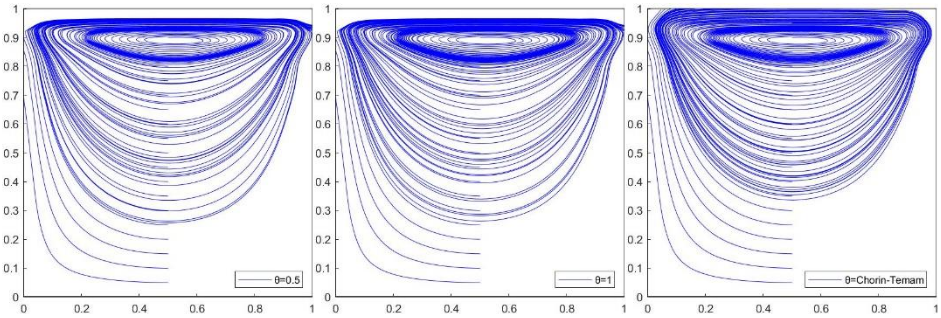

Figure 3. Streamlines for Time=0.2s

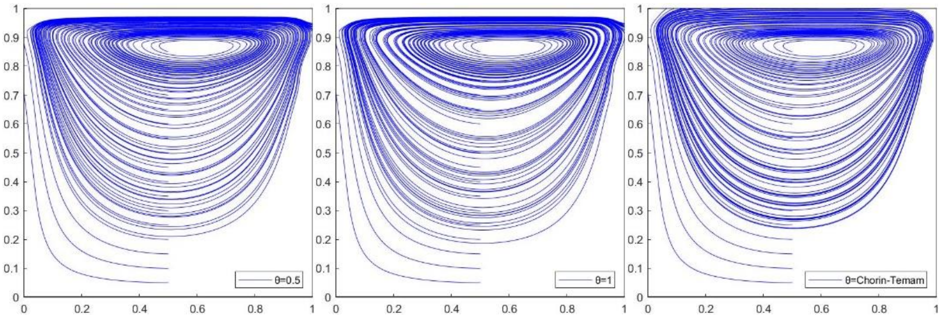

Figure 4. Streamlines for Time=0.5s

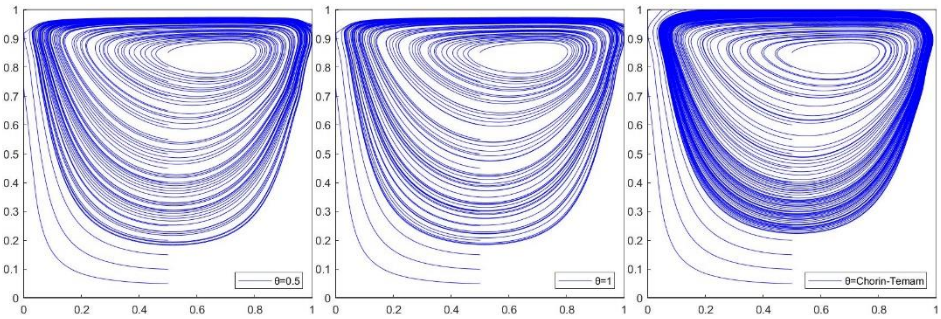

Figure 5. Streamlines for Time=1s

Obviously, all the results are very similar in every time-step. The semi-implicit method computes the flow having reached further depth of the cavity while $T = 1\text{s}$. In semi-implicit, $\mathrm{y} \approx 0.6$, while in Chorin-Temam $\mathrm{y} \approx 0.5$. This could be affected by the different viscous and convective term on both methods.

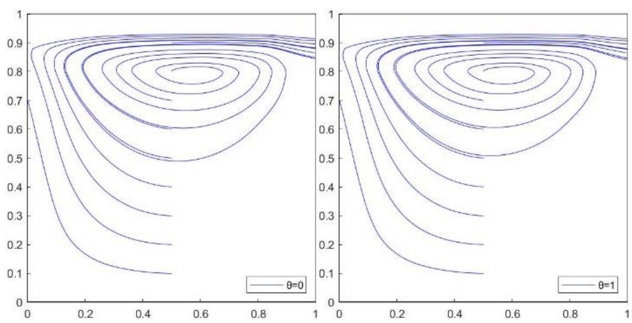

Figure 6. Streamlines for Time=1s

While $\theta < 0.5$, the semi-implicit method can only converge with SUPG stabilization method by Q1Q1 elements. We can find the results in Figure 6 are also looks similar comparing $\theta = 0$ and $\theta = 1$.

4. REFERENCE

[1] Lecture slides in Finite elements in fluid.
[2] Finite Element Methods for Flow Problems, Jean Donea and Antonio Huerta.
[3] http://www.scielo.br/scielo.php?script=sci_arttext&pid=S1678-58782009000300004