International Centre for
Numerical Methods in Engineering

Universitat Politècnica de Catalunya

Master of Science in Computational Mechanics

Finite Element in Fluids

Homework 6A - Stokes Problem

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1 Cavity Stokes Problem

Problem Statement

Consider the governing equations that describe a highly viscous isotropic incompressible flow (with low Reynolds number), known as *Stokes Equations*:

$$\begin{cases} -\nu\nabla^2\mathbf{v} + \nabla p = \mathbf{b} & in \quad \Omega \\ \nabla\cdot\mathbf{v} = 0 & in \quad \Omega \end{cases} \tag{1.1}$$

Where:

- $\mathbf{v}$ - is the velocity.
- p - is the pressure.
- ν - is kinematic viscosity.
- $\mathbf{b}$ - is a source term.

1.1 Cavity Flow

The cavity flow problem is the benchmark used to test the stabilization terms in this work. The figure 1.1 shows the geometry of the problem and includes the boundary conditions which presents zero velocities in each wall except for the upper wall, which has a magnitude of 1 in the positive horizontal direction. Because of that, the solution of the streamlines will behave as a vortex inside the domain. The pressure value is equal to zero in all the sides, but it should be considered that at the two upper corners (where the velocity boundary condition is applied) there will appear a singularity in pressure in each corner.

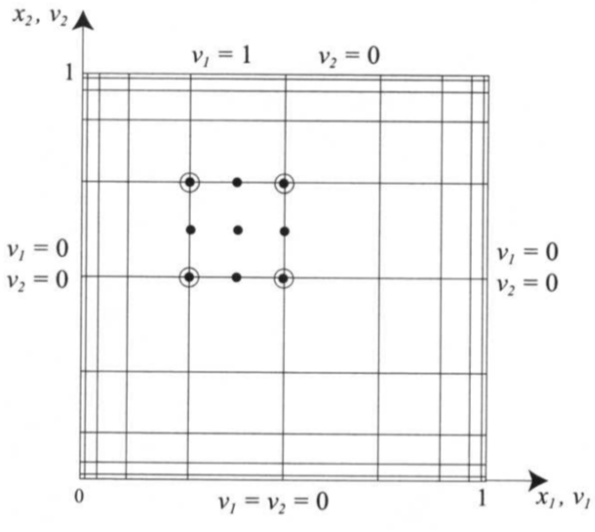
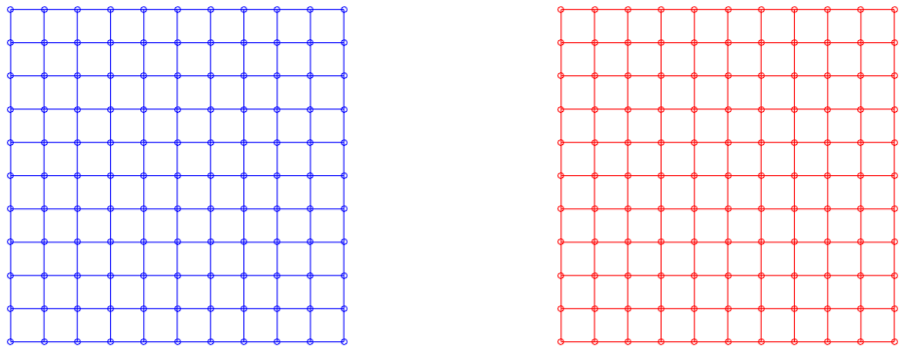


Figure 1.1: Cavity Flow Problem boundary conditions.

1.2 Discretization Comparison

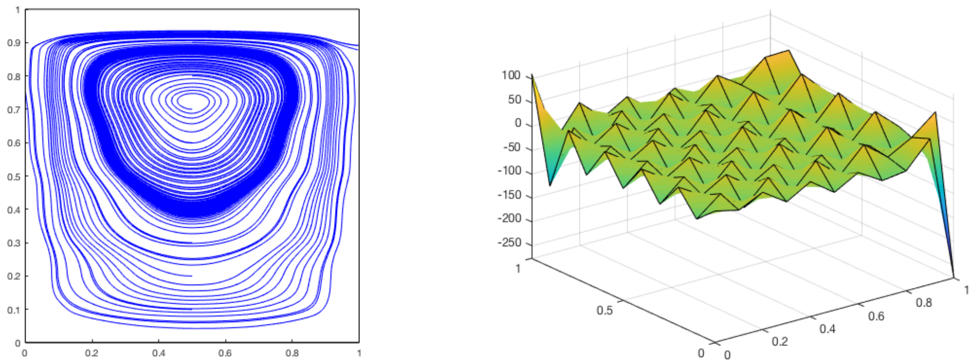
The code provided in class allows to discretize the domain using different types of elements as quadrilateral and triangular, and also permits to change the order of approximation, which can be linear or quadratic. Then, in order to verify how stable and accurate are the solutions, a comparison between elements is performed. The chosen mesh is 10x10 quadrilateral elements which corresponds to the double elements for triangles.

- **Quadrilateral Element Q1Q1:** The elements discretization for velocity and pressure are shown in the figure 1.2. Then it can be seen in the resulting graph of pressure solution (figure 1.3b), that there is instabilities due the linear numerical approximation, for that reason *does not fulfill the LBB condition.* Its steramlines are showed in graph 1.3a.
- **Quadrilateral Element Q2Q1:** Now consider the discretization shown in the figure 1.4. Now, the pressure response is *stable* as it can be seen in the graph 1.5b. The corresponding velocity solution is more accurate which is depicted in figure 1.5a.
- **Triangular Element P1P1:** This test is considering the same spatial divisions but with the double of elements as it is seen in 1.6. Similarly to the linear quadrilateral, the triangular element is *not LBB stable* as can be observed in the pressure solution (fig 1.7b).
- **Triangular Element P2P1:** Finally the discretization for quadratic velocity and linear pressure is shown in 1.8. As expected the solution is as *stable* as quadrilaterals with minimal differences in the approximated solution. (Consider graphs 1.9b and 1.9a).



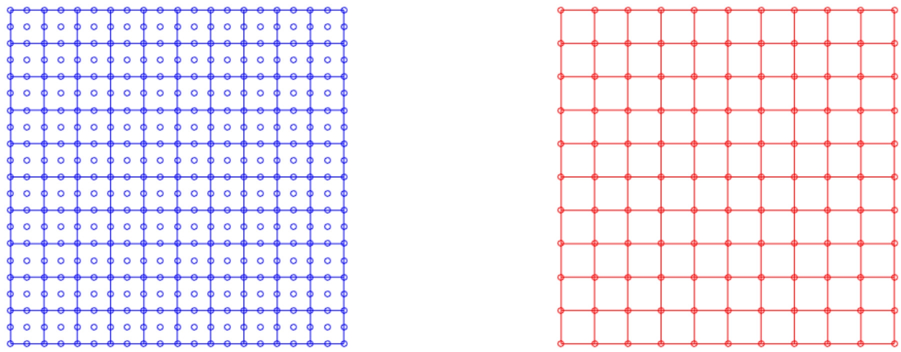
(a) (b)

Figure 1.2: a) Velocity and b) pressure discretization for quadrilateral Q1Q1.



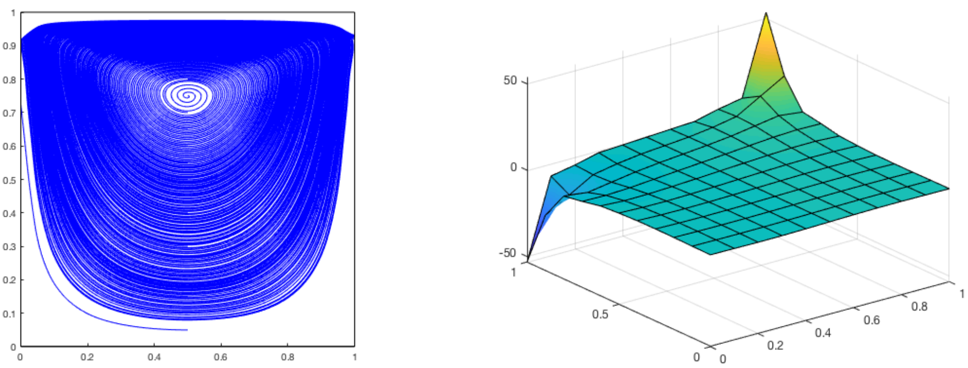
(a) (b)

Figure 1.3: a) Velocity and b) pressure solutions for quadrilateral Q1Q1.



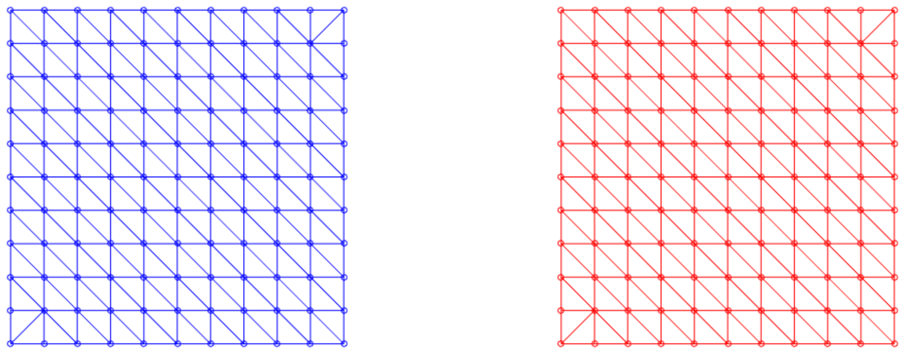
(a) (b)

Figure 1.4: a) Velocity and b) pressure discretization for quadrilateral Q2Q1.



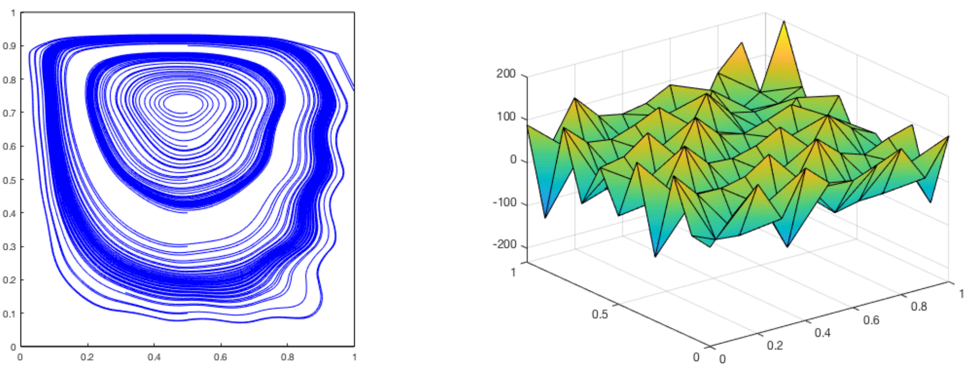
(a) (b)

Figure 1.5: a) Velocity and b) pressure solutions for quadrilateral Q2Q1.



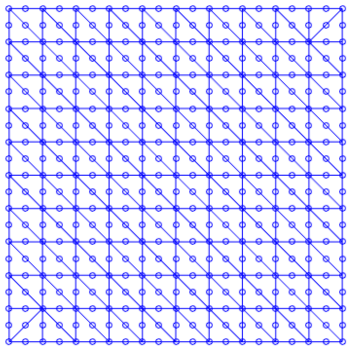
(a) (b)

Figure 1.6: a) Velocity and b) pressure discretization for Triangular P1P1.

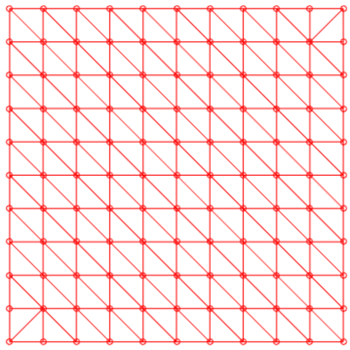


(a) (b)

Figure 1.7: a) Velocity and b) pressure solutions for Triangular P1P1.

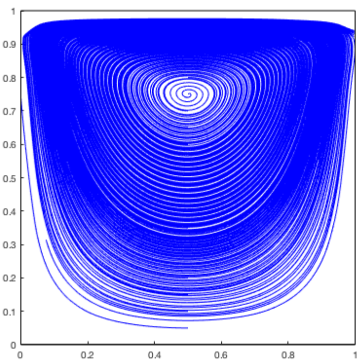


(a)

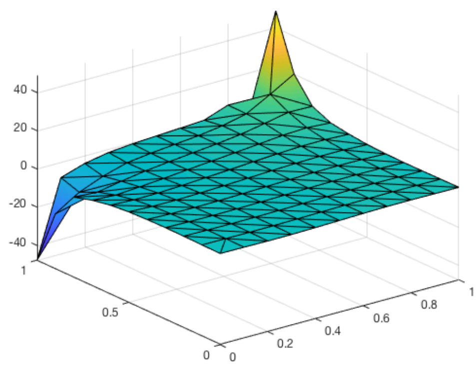


(b)

Figure 1.8: a) Velocity and b) pressure discretization for Triangular P2P1.



(a)



(b)

Figure 1.9: a) Velocity and b) pressure solutions for Triangular P2P1.

1.3 Stabilization Method

In order to obtain stable solutions, it can be applied a **GLS Stabilization Method** into the elemental matrices solved before. Similar than the 1D problem of convection, GLS employs the residual to add a stabilization term that provides more symmetry to the system of equations matrix. The difference with the Stokes problem in 2D is that the implementation is more involved and requires carefully attention of the resulting expressions. If Stokes problem can be written as $\mathcal{L}(\mathbf{v}, p) = \mathcal{F}$ where:

$$\mathcal{L}(\mathbf{v}, p) = \begin{bmatrix} -\nu\nabla^2\mathbf{v} + \nabla p \\ \nabla \cdot \mathbf{v} \end{bmatrix} \tag{1.2}$$

$$\mathcal{F} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix} \tag{1.3}$$

The weak form is then:

$$\int_\Omega \begin{bmatrix} \mathbf{w} \\ q \end{bmatrix} \cdot \big(\mathcal{L}(\mathbf{v}, p) - \mathcal{F}\big)\, d\Omega = 0 \tag{1.4}$$

Now, the GLS stabilization term is:

$$\sum_e \int_{\Omega_e} \tau \mathcal{L}(\mathbf{v}, p) \cdot \big(\mathcal{L}(\mathbf{v}, p) - \mathcal{F}\big)\, d\Omega = 0 \tag{1.5}$$

In a practical view, before the system of equations needed to solve have the form:

$$\begin{pmatrix} \mathbf{K} & \mathbf{G}^T \\ -\mathbf{G} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{h} \end{pmatrix} \tag{1.6}$$

After performing the discretization of the stabilized terms the new system of equations is:

$$\begin{pmatrix} \mathbf{K} + \bar{\mathbf{K}} & \mathbf{G}^T + \bar{\mathbf{G}}^T \\ -\mathbf{G} + \bar{\mathbf{G}} & \mathbf{0} + \bar{\mathbf{L}} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} + \bar{\mathbf{f}}_\mathbf{w} \\ \mathbf{h} + \bar{\mathbf{f}}_\mathbf{q} \end{pmatrix} \tag{1.7}$$

It can be proved that some of the stabilization terms added after perform the discretization need a second order derivative. So, in a general implementation this would be needed to consider but as in this work the instabilities shown are only in the linear elements Q1Q1 and P1P1, then the only terms to code are $\bar{\mathbf{L}}$ and $\bar{\mathbf{f}}_\mathbf{q}$ because the other will be zero at performing the second derivative. The expressions are:

$$\bar{\mathbf{L}} = \sum_e \int_{\Omega_e} \tau_1 (\nabla q) \cdot (\nabla p)\, d\Omega_e \tag{1.8}$$

$$\bar{\mathbf{f}}_\mathbf{q} = \sum_e \int_{\Omega_e} \tau_1 (\nabla q) \cdot (-f)\, d\Omega_e \tag{1.9}$$

In that sense, the implementation in Matlab can be done by using parts of the original code because the discretization of those terms can be seen respectively as:

$$(\nabla q)\cdot(\nabla p) = \left[\begin{array}{cc} \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{array}\right]\left[\begin{array}{c} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \end{array}\right] \tag{1.10}$$

$$(\nabla q)\cdot(-f) = \left[\begin{array}{cc} \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} \end{array}\right]\left[\begin{array}{c} -f_1 \\ -f_2 \end{array}\right] \tag{1.11}$$

The vector τ to guarantee convergence and stability is:

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\left(\frac{h^2}{4\nu}\right) \\ 0 \end{bmatrix} \tag{1.12}$$

Code Implementation 1

The only lines to add in the file *StokesSystem.m* are the computation of τ_1 based on the size of the element and the corresponding submatrix components of $\bar{\mathbf{L}}$ and $\bar{\mathbf{f}}_{\mathbf{q}}$. Then:

```
h=XP(2)-XP(1);
tau1 = 1/3*h^2/(4*mu);
```

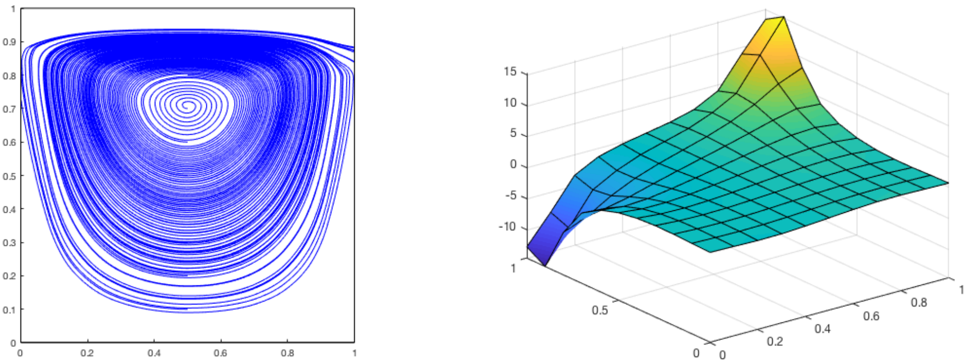
And inside the for loop of the Gauss points:

```
Le = Le - tau1*(nx'*nx+ny'*ny)*dvolu;
f_qe = f_qe - tau1*([nx; ny]'*f_igaus)*dvolu;
```

It is necessary to modify the parameters of the *StokesSystem.m* in order to pass to the *mainStokes.m* the matrices and inside the last file, a modification in the complete system is:

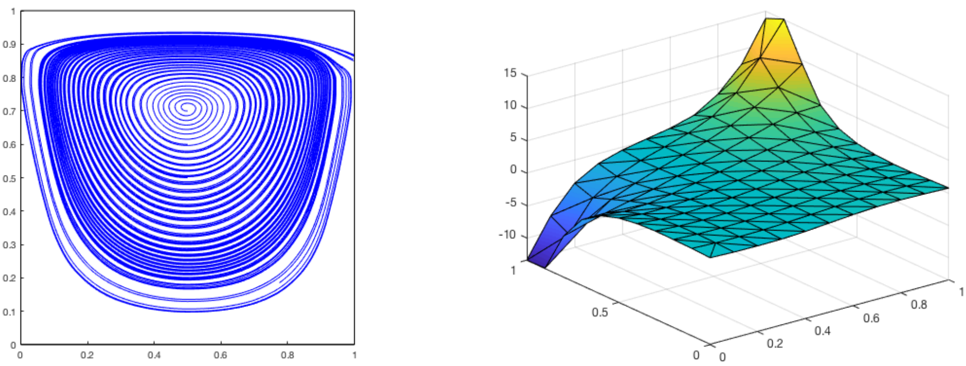
```
    A = [Kred    Gred';
         Gred    L];
    b = [fred; f_q];
```

Finally, the results of the stabilized implementations are shown in the next graphs. Therefore, the stabilized quadrilateral element Q1Q1 velocity and pressure solutions coincide with the stabilized triangular element P1P1, there is no oscillations or spurious solutions in the pressure plot, and the velocity streamlines coincide with the higher order elements.



(a) (b)

Figure 1.10: a) Velocity and b) pressure solutions for stabilized Q1Q1.



(a) (b)

Figure 1.11: a) Velocity and b) pressure solutions for stabilized Q1Q1.