Màster en Mètodes Numèrics **Aitor Bazán Escoda**

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Finite Elements in Fluids

<u>Homework 6: Unsteady Navier-Stokes</u>

1. Introduction

Considering the case of unsteady viscous and incompressible flows, the governing equations regarding such fluid flow are stated as follows.

$$\begin{cases} \mathrm{v_t} - \nu\nabla^2\mathrm{v} + (\mathrm{v}\cdot\nabla)\mathrm{v} + \nabla\mathrm{p} = \mathrm{b} & \text{in } \Omega \\ \mathrm{v}\cdot\nabla = 0 & \text{in } \Omega \\ \mathrm{v} = \mathrm{v_D} & \text{on } \Gamma_\mathrm{D} \\ -\mathrm{pn} + \nu(\mathrm{n}\cdot\nabla)\mathrm{v} = \mathrm{t} & \text{on } \Gamma_\mathrm{N} \\ \mathrm{v}(\mathrm{x},0) = \mathrm{v}_{0(\mathrm{x})} & \text{in } \Omega \end{cases} \tag{1}$$

Then, unsteady Navier-Stokes equations are composed for the unsteady term, the viscous term, the convective term, pressure gradient and external body forces.

Initial velocity field is assumed solenoidal, $\nabla\cdot\boldsymbol{v_0} = 0$.

The model used to apply the transient problem is a squared, $[0,1]^2$ cavity with flow inside and an upper driven lid whose prescribed velocity is $v^*{}_x = 1$, while all the other walls are fixed. Dirichlet boundary conditions are described below, although were already explained on *Hw5: Incompressible flow. Stokes and Navier-Stokes.*
Note that, since only dirichlet bc are imposed, just relative pressure fields are obtained, thus, confined pressure is prescribed in the lower left corner of the cavity with a value of P=0. Moreover, kinematic viscosity is assumed to be $\nu = 0{,}01$ and Reynolds $Re = 100$.

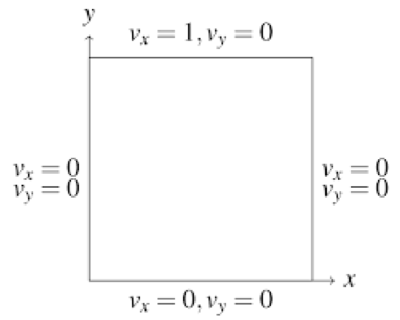

Figure 1. Lid cavity flow problem.

Equations in (1) are projected onto a space of weighting functions such that the wear formuation is obtained for the momentum equation. Therefore, Galerkin spatial discretization of the transient problem yields into the system of semi-discrete equations (2).

$$\begin{cases} \mathbf{M}\dot{\mathbf{u}}(\mathrm{t}) + \left[\mathbf{K} + \mathbf{C}\big(\mathrm{v}(\mathrm{t})\big)\right]\mathbf{u}(\mathrm{t}) + \mathbf{G}\mathbf{p}(\mathrm{t}) = \mathbf{b} \\ \mathbf{G}^\mathbf{T}\mathbf{u}(\mathrm{t}) = 0 \\ \mathbf{u}(0) = \mathbf{v}_0 - \mathbf{v}_\mathrm{D}(0) \end{cases} \tag{2}$$

where **M** is the standard finite element mass matrix.

The advancing in time for the above system of equations are discretized, for the current case, with a semi-implicit method and the *Chorin-Temam*.

$$\boldsymbol{v}_t = w \cdot \frac{\partial}{\partial t}\begin{bmatrix} v_x \\ v_y \end{bmatrix} \rightarrow \boldsymbol{M} = \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \end{bmatrix}\begin{bmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{bmatrix} \tag{3}$$

where w is the weighting function.

2. Semi-implicit discretization

So as to represent the semi-implicit method presented within the report, *Theta method* is used for the time discretization [Eq. 4].

$$\frac{\Delta \mathbf{u}}{\Delta \mathrm{t}} - \theta \Delta \mathbf{u}_{\mathrm{t}} = \mathbf{u}_{\mathrm{t}}^{\mathrm{n}} \tag{4}$$

being $\Delta u = u^{n+1} - u^n, n$ the time step, and θ denoting if the method used will be implicit or explicit.

If we depart from the substitution of the first partial differential equation shown on the set of equations 2, it is obtained that,

$$\mathbf{u_t} = \frac{\mathbf{b} - \big(\mathbf{K} + \mathbf{C}(\mathrm{v})\big)\mathbf{u} - \mathbf{G}\mathbf{p}}{\mathbf{M}} \tag{5}$$

therein, the substitution of [Eq. 5] into [Eq. 4] yields,

$$\frac{\Delta \mathbf{u}}{\Delta \mathrm{t}} - \theta \left(\frac{-(\mathbf{K} + \mathbf{C}^{\mathrm{m}+1})\mathbf{u}^{\mathrm{n}+1} + (\mathbf{K} + \mathbf{C}^{\mathrm{m}})\mathbf{u}^{\mathrm{n}} - \mathbf{G}\mathbf{p}^{\mathrm{n}+1} + \mathbf{G}\mathbf{p}^{\mathrm{n}}}{\mathbf{M}} \right) = \frac{\mathbf{b} - (\mathbf{K} + \mathbf{C}^{\mathrm{m}})\mathbf{u}^{\mathrm{n}} - \mathbf{G}\mathbf{p}^{\mathrm{n}}}{\mathbf{M}} \tag{6}$$

In the Equation 6, following the semi-implicit method, the system is solved implicitly. However, the convective term is treated explicitly. Thus, C^{m+1} is evaluated as C^m, becoming, Equation 6 as,

$$\frac{\Delta \mathbf{u}}{\Delta \mathrm{t}} + \theta \frac{(\mathbf{K} + \mathbf{C}^{\mathrm{m}})\Delta \mathbf{u} + \mathbf{G}\Delta \mathbf{p}}{\mathbf{M}} = \frac{\mathbf{b} - (\mathbf{K} + \mathbf{C}^{\mathrm{m}})\mathbf{u}^{\mathrm{n}} - \mathbf{G}\mathbf{p}^{\mathrm{n}}}{\mathbf{M}} \tag{7}$$

if terms are rearranged, from Equation 7, it yields:

$$\big(\mathbf{M} + \theta \Delta \mathrm{t}(\mathbf{K} + \mathbf{C}^{\mathrm{m}})\big)\Delta \mathbf{u} + \Delta \mathrm{t}\theta \mathbf{G}\Delta \mathbf{p} = \Delta \mathrm{t}(\mathbf{b} - (\mathbf{K} + \mathbf{C}^{\mathrm{m}})\mathbf{u}^{\mathrm{n}} - \mathbf{G}\mathbf{p}^{\mathrm{n}}) \tag{8}$$

However, because of incompressibility condition, it implies free divergence at all time steps. Therefore, working out the final system of equations, it reads as,

$$\begin{cases} \big(\mathbf{M} + \theta \Delta \mathrm{t}(\mathbf{K} + \mathbf{C}^{\mathrm{m}})\big)\Delta \mathbf{u} + \Delta \mathrm{t}\theta \mathbf{G}\Delta \mathbf{p} = \Delta \mathrm{t}(\mathbf{b} - (\mathbf{K} + \mathbf{C}^{\mathrm{m}})\mathbf{u}^{\mathrm{n}} - \mathbf{G}\mathbf{p}^{\mathrm{n}}) \\ \mathbf{G}^{\mathrm{T}}\Delta \mathbf{u} = 0 \end{cases} \tag{9}$$

which in matrix form, and providing the increment of the solution at each time step, is as follows,

$$\begin{bmatrix} \boldsymbol{M} + \theta \Delta t(\boldsymbol{K} + \boldsymbol{C}^m) & \Delta t \theta \boldsymbol{G} \\ \boldsymbol{G}^T & 0 \end{bmatrix}\begin{bmatrix} \Delta \boldsymbol{u} \\ \Delta \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \Delta t(\boldsymbol{b} - (\boldsymbol{K} + \boldsymbol{C}^m)\boldsymbol{u}^n - \boldsymbol{G}\boldsymbol{p}^n) \\ 0 \end{bmatrix} \tag{10}$$

2.1. Results

The domain is discretized into a 10x10 staggered mesh of Q2Q1 elements. Semi-implicit method with $\theta = 1$ and $\theta = 0.5$ are used with a time step value of $\Delta t = 0.01s$. Moreover, here below are plotted the solution for those theta methods at different time steps (t=0.25, so as to show the behaviour of the flow at different stages of the solution.

2.1.1. Theta method, $\boldsymbol{\theta = 0.5}$.

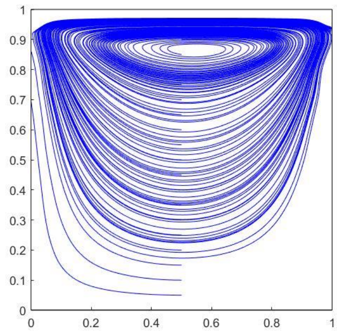

Figure 2. Crank-Nicolson streamlines at t=0.25s.

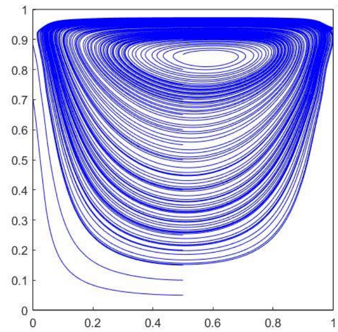

Figure 3. Crank-Nicolson streamlines at t=0.5s.

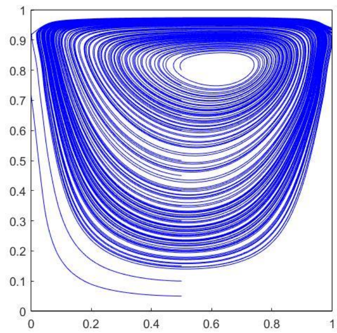

Figure 4. Crank-Nicolson streamlines at t=0.25s.

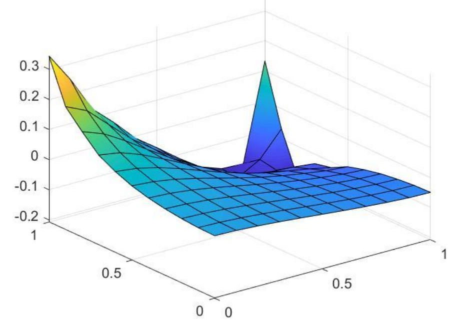

Figure 5. Crank-Nicolson Press. solution at t=1s.

As clearly seen in *Fig.* , solution for $\theta < 0.5$ does not converge. So, it is used a basic SUPG stabilization technique (Q1Q1 element type) and solution finally did converge.

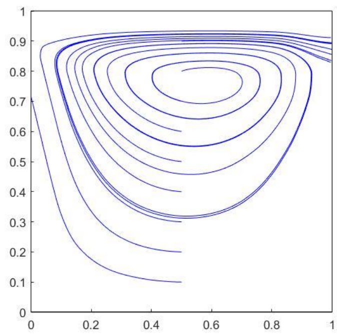

Figure 6. Crank-Nicolson streamlines at t=1s. Q1Q1 element type.

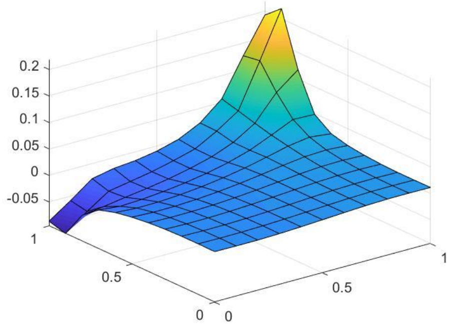

Figure 7. Crank-Nicolson Press. solution at t=1s. Q1Q1 element type.

2.1.2. Theta method, $\theta = 1$.

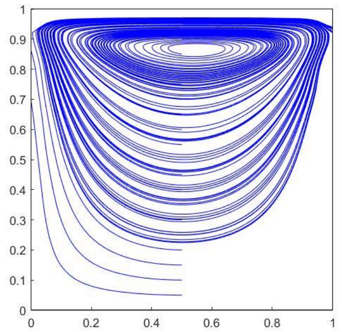

Figure 8. Implicit method streamlines at t=0.25s.

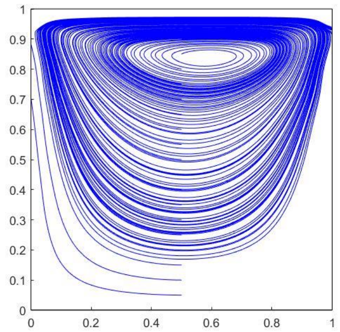

Figure 9. Implicit method streamlines at t=0.5s.

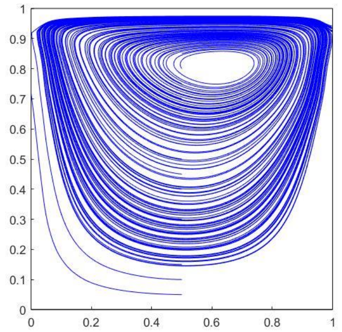

Figure 10. Implicit method streamlines at t=1s.

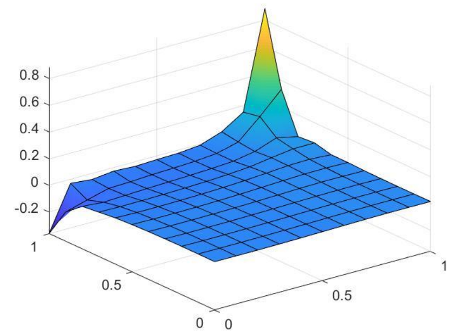

Figure 11. Implicit method Press. solution at t=1s.

In order to show reliability on results for $\theta < 0.5$, it is compared with same element type the similarity in the convergence of the results.

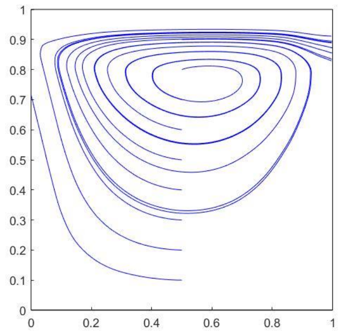

Figure 12. Implicit method streamlines at t=1s. Q1Q1 element type.

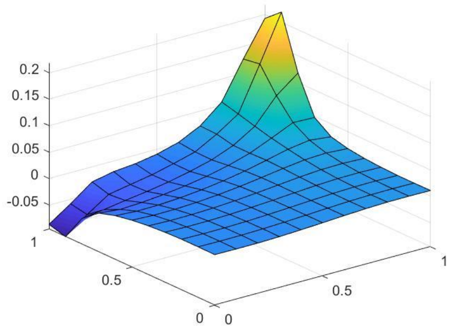

Figure 13. Implicit method Press. solution at t=1s. Q1Q1 element type.

3. Chorin-Temam projection method

The principle of the *Chorin-Temam* projection method is to compute the velocity and pressure fields separately throughout two stages. In the first one, an intermediate velocity that does not satisfy the incompressibility constraint is computed at each time step. In the second stage, the pressure is used to project that intermediate velocity onto a space of divergence-free velocity field in order to get the next updates for pressure and velocity.

The development of the discretization for the first stage of the method is driven first through the computation of the intermediate velocity $\boldsymbol{u}^*$ explicitly using the momentum equation and ignoring the pressure gradient. Thus,

$$\boldsymbol{M}\boldsymbol{u}_t + (\boldsymbol{K} + \boldsymbol{C})\boldsymbol{u} = \boldsymbol{b} \tag{11}$$

which in fact, by rearranging terms and solving at each time step is,

$$\big(\boldsymbol{M} + \Delta t(\boldsymbol{K} + \boldsymbol{C}^m)\big)\boldsymbol{u}^* = \boldsymbol{M}\boldsymbol{u}^n + \Delta t\boldsymbol{b} \tag{12}$$

The second stage, within the projection step, starts with the intermediate velocity being corrected to obtain the final solution of time step $\boldsymbol{u}^{n+1}$. Then,

$$\boldsymbol{M}\boldsymbol{u}_t + \boldsymbol{G}\boldsymbol{p} = 0 \tag{13}$$

$$\begin{cases} \boldsymbol{M}\boldsymbol{u}^{n+1} + \Delta t\boldsymbol{G}\boldsymbol{p}^{n+1} = \boldsymbol{M}\boldsymbol{u}^* \\ \qquad\quad \mathbf{G}^{\mathbf{T}}\mathbf{u}^{\boldsymbol{n}+\mathbf{1}} = 0 \end{cases} \tag{14}$$

which in matrix form leads to the following,

$$\begin{bmatrix} \boldsymbol{M} & \Delta t\boldsymbol{G} \\ \boldsymbol{G}^T & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}^{n+1} \\ \boldsymbol{p}^{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{M}\boldsymbol{u}^* \\ 0 \end{bmatrix} \tag{15}$$

3.1. Results

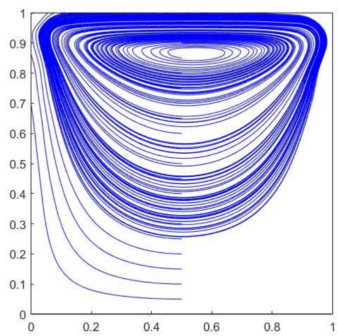

Figure 14. Chorin-Temam streamlines at t=0.25s.

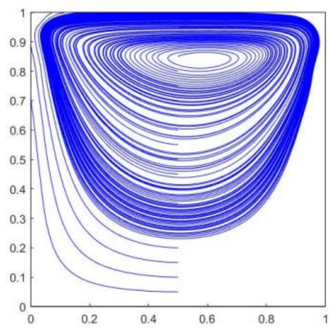

Figure 15. Chorin-Temam streamlines at t=0.5s.

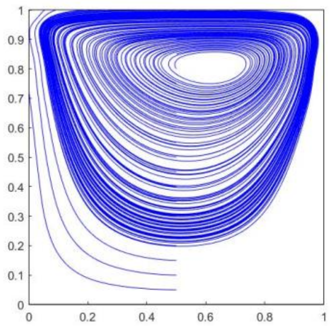

Figure 16. Chorin-Temam streamlines at t=1s.

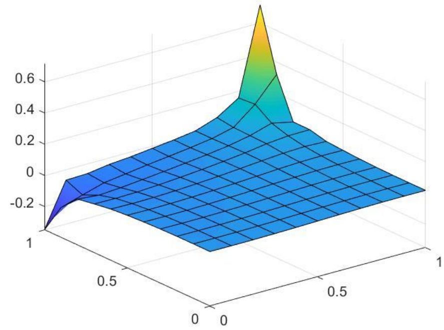

Figure 17. Chorin-Temam Press. solution at t=1s.

A fully implicit method requires the solution of a nonlinear system of equations at each time step. Though, semi-implicit methods are preferred since convective term and external forces are treated explicitly.