FEF-HW5: Stokes and Navier-Stokes equations

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1 Stokes problem

The **strong form** of the problem is:

$$
\begin{aligned}
-\nu\nabla^2\mathbf{v} + \nabla p &= \mathbf{b} \;\text{ in } \Omega \\
\nabla\cdot\mathbf{v} &= 0 \;\text{ in } \Omega
\end{aligned}
$$

Considering $\omega \in \mathcal{V}$ and $q \in \mathcal{Q}$, we end with the **weak form**:

$$
\begin{aligned}
\int_\Omega \nabla\omega : \nu\nabla\mathbf{v}\, d\Omega - \int_\Omega p\nabla\cdot\omega\, d\Omega &= \int_\Omega \omega\cdot\mathbf{b}\, d\Omega \\
\int_\Omega q\nabla\cdot\mathbf{v}\, d\Omega &= 0
\end{aligned}
$$

Considering $\{N_i(\mathbf{x})\}$ a base of $\mathcal{V}$ and $\{\tilde{N}_i(\mathbf{x})\}$ a base of $\mathcal{Q}$, we approximate our functions by

$$
\begin{aligned}
\mathbf{v} \approx \mathbf{v}^h &= \sum_j \mathbf{v_j} N_j(\mathbf{x}) \\
p \approx p^h &= \sum_k p_k \tilde{N}_k(\mathbf{x})
\end{aligned}
$$

and we define the auxiliar functions ω and q by

$$
\begin{aligned}
\omega &= \sum_i \omega_{\mathbf{i}} N_i(\mathbf{x}) \\
q &= \sum_l q_l \tilde{N}_l(\mathbf{x})
\end{aligned}
$$

Substituting in the weak form,

$$
\begin{aligned}
\mathbf{W}^T\int_\Omega (grad\,\mathbf{N})^T\nu(grad\,\mathbf{N})\, d\Omega\, \mathbf{V} - \mathbf{W}^T\int_\Omega \mathbf{D}^T(mat\,\tilde{\mathbf{N}})\, d\Omega\, \mathbf{P} &= \mathbf{W}^T\int_\Omega (mat\,\mathbf{N})\cdot\mathbf{b}\, d\Omega \\
\mathbf{Q}^T\int_\Omega (mat\,\tilde{\mathbf{N}})^T\mathbf{D}\, d\Omega\, \mathbf{V} &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
(grad\,\mathbf{N}) &= \begin{pmatrix} \frac{\partial N_1}{\partial x} & 0 & \dots & \frac{\partial N_m}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial x} & \dots & 0 & \frac{\partial N_m}{\partial x} \\ \frac{\partial N_1}{\partial y} & 0 & \dots & \frac{\partial N_m}{\partial y} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & \dots & 0 & \frac{\partial N_m}{\partial y} \end{pmatrix} \\
(mat\,\mathbf{N}) &= \begin{pmatrix} N_1 & 0 & \dots & N_m & 0 \\ 0 & N_1 & \dots & 0 & N_m \end{pmatrix} \\
(mat\,\tilde{\mathbf{N}}) &= \begin{pmatrix} \tilde{N_1} & \tilde{N_2} & \dots & \tilde{N_m} \end{pmatrix} \\
(grad\,\tilde{\mathbf{N}}) &= \begin{pmatrix} \frac{\partial \tilde{N_1}}{\partial x} & \frac{\partial \tilde{N_2}}{\partial x} & \dots & \frac{\partial \tilde{N_m}}{\partial x} \\ \frac{\partial \tilde{N_1}}{\partial y} & \frac{\partial \tilde{N_2}}{\partial y} & \dots & \frac{\partial \tilde{N_m}}{\partial y} \end{pmatrix} \\
\mathbf{D} &= [1,0,0,1](grad\,\mathbf{N}) \qquad (1) \\
\mathbf{V}^T &= [V_{x1}, V_{y1}, ...V_{xm}, V_{ym}] \qquad (2) \\
\mathbf{P}^T &= [P_1, P_2, ...P_{m-1}, P_m] \qquad (3)
\end{aligned}
$$

Changing the sign of the last equation we end with the linear system:

$$
\begin{pmatrix} \mathbf{K} & \mathbf{G}^T \\ \mathbf{G} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{V} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix} \qquad (4)
$$

where

$$
\begin{aligned}
\mathbf{K} &= \int_\Omega [grad\,\mathbf{N}]^T \nu [grad\,\mathbf{N}]\, d\Omega \\
\mathbf{G} &= -\int_\Omega [mat\,\tilde{\mathbf{N}}]^T D\, d\Omega \\
\mathbf{f} &= \int_\Omega [mat\,\mathbf{N}] \cdot \mathbf{b}\, d\Omega
\end{aligned}
$$

1.1 Results

Considering different basis for the two spaces $\mathcal{V}$ and $\mathcal{Q}$, we obtain the following results:

Type of element for $\mathcal{V}$	Q1	Q2	P1	P2
Type of element for $\mathcal{Q}$	Q1	Q1	P1	P1
Stable?	No	Yes	No	Yes

Table 1: Where Q denotes a quadrilateral element and P a triangle element, and the number specifies the degree of the functions $N/\tilde{N}$.

As we can see, we need to stabilize the results for linear element for $\mathcal{V}$.

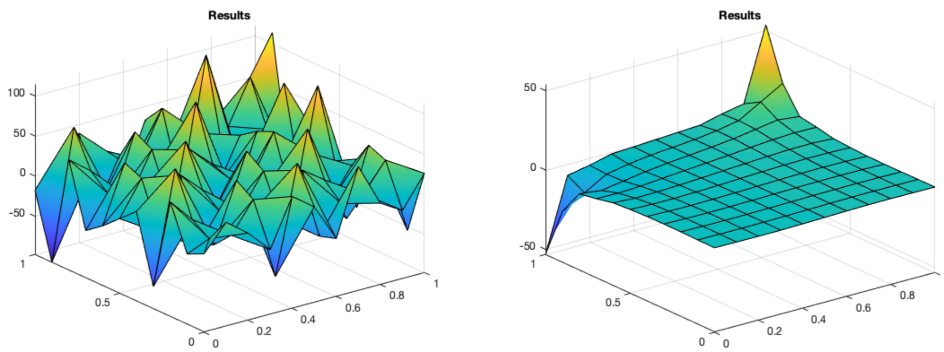

Figure 1: Examples of the resulting pressure. On the left, results for P1P1 elements, and on the right, results for Q2Q1 elements.

1.2 Stabilization method (GLS)

We consider the GLS stabilization method which consists in adding stabilization terms to the weak form:

$$\sum_e \int_{\Omega_e} \tau \mathcal{L}(\omega, q)(\mathcal{L}(\mathbf{v}, p) - \mathcal{F})\, d\Omega \tag{5}$$

where

$$\mathcal{L}(\mathbf{v}, p) = \begin{bmatrix} -\nu \nabla^2 \mathbf{v} + \nabla p \\ \nabla \cdot \mathbf{v} \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} \mathbf{f} \\ 0 \end{bmatrix}$$

Adding the term to the weak form and using that we are only stabilizing for linear element in velocity and in pressure (so $\nabla^2 \mathbf{v} = \nabla^2 \omega = 0$), we end with the system

$$\begin{pmatrix} \mathbf{K} + \bar{\mathbf{K}} & \mathbf{G}^T \\ \mathbf{G} & \bar{\mathbf{L}} \end{pmatrix} \begin{pmatrix} \mathbf{V} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \bar{\mathbf{f}_\mathbf{q}} \end{pmatrix}$$

where

$$\begin{aligned} \bar{\mathbf{K}} &= \sum_e \int_{\Omega_e} \tau_2 \mathbf{D}^T \mathbf{D}\, d\Omega \\ \bar{\mathbf{L}} &= \sum_e \int_{\Omega_e} \tau_1 (grad\, \tilde{\mathbf{N}})^T (grad\, \tilde{\mathbf{N}})\, d\Omega \\ \bar{\mathbf{f}_\mathbf{q}} &= \sum_e \int_{\Omega_e} \tau_1 (grad\, \tilde{\mathbf{N}})^T \cdot \mathbf{b}\, d\Omega \end{aligned}$$

We will take $\tau_2 = 0$ and $\tau_1 = \frac{h^2}{12\nu}$.

I do not get the expected results.

1.3 Description of the problem (BC)

For the velocity, we are imposing Dirichlet conditions along $\delta\Omega = \delta([0,1]^2)$: $\mathbf{v} = (0,0)$ on $x = 0$, $x = 1$ and $y = 0$ and $\mathbf{v} = (1,0)$ on $y = 1$.

For the pressure, we are imposing natural boundary conditions ($\nabla p \cdot \mathbf{n} = 0$).

This is physically equivalent to a sink, since we are inducing a rotation along the fluid, increasing the pressure in the top-right corner and decreasing it in the top-left corner.

2 Navier-Stokes problem

In this case, the **strong form** of the problem is:

$$\begin{aligned} -\nu \nabla^2 \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p &= \mathbf{b} \ \text{ in } \Omega \\ \nabla \cdot \mathbf{v} &= 0 \ \text{ in } \Omega \end{aligned}$$

Thus we will have to add a convective term to the linear system 4, $\mathbf{C}(\mathbf{v}) = \int_\Omega (mat\,\mathbf{N})\mathbf{V}\mathbf{D}\, d\Omega$, since

$$\mathbf{C}(\bar{\mathbf{v}})\mathbf{v} = \begin{pmatrix} \bar{v}_x \frac{\partial}{\partial x} v_x + \bar{v}_x \frac{\partial}{\partial y} v_y \\ \bar{v}_y \frac{\partial}{\partial x} v_x + \bar{v}_y \frac{\partial}{\partial y} v_y \end{pmatrix} = \begin{pmatrix} \bar{v}_x \\ \bar{v}_y \end{pmatrix} \begin{pmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & \dots & \frac{\partial N_m}{\partial x} & \frac{\partial N_m}{\partial y} \end{pmatrix} \mathbf{V} = (mat\,\mathbf{N})\bar{\mathbf{V}}\mathbf{D}\mathbf{V}$$

Thus we end up with the following nonlinear system:

$$\begin{pmatrix} \mathbf{K} + \mathbf{C}(\mathbf{v}) & \mathbf{G}^T \\ \mathbf{G} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{V} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix} \tag{6}$$

To solve the system, we will use two methods: Picard's method and Newton method.

2.1 Picard's method

In this case: $\Delta \mathbf{x}^{k+1} = F^{-1}(\mathbf{x}^{k+1})(\mathcal{F} - F(\mathbf{x}^k)\mathbf{x}^k)$, where $\mathbf{x} = (\mathbf{V}, \mathbf{P})$ and we are considering eq.6 as $F(\mathbf{x})\mathbf{x} = \mathcal{F}$.

2.2 Newton's method

In this case: $\Delta \mathbf{x}^{k+1} = -\mathbf{J}(\mathbf{x}^k)(F(\mathbf{x}^k)\mathbf{x}^k - \mathcal{F})$ where $\mathbf{J}(\mathbf{x}^k)$ is the Jacobian matrix of the system. We know that

$$\begin{aligned} \mathbf{C}(\bar{\mathbf{v}})\mathbf{v} &= \begin{pmatrix} \bar{v}_x \frac{\partial}{\partial x} v_x + \bar{v}_x \frac{\partial}{\partial y} v_y \\ \bar{v}_y \frac{\partial}{\partial x} v_x + \bar{v}_y \frac{\partial}{\partial y} v_y \end{pmatrix} \\ &= \begin{pmatrix} \bar{v}_x \frac{\partial}{\partial x} & \bar{v}_x \frac{\partial}{\partial y} \\ \bar{v}_y \frac{\partial}{\partial x} & \bar{v}_y \frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y & 0 \\ 0 & \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y \end{pmatrix} \begin{pmatrix} \bar{v}_x \\ \bar{v}_y \end{pmatrix} = (\nabla \cdot \mathbf{v}) Id \bar{\mathbf{v}} \end{aligned}$$

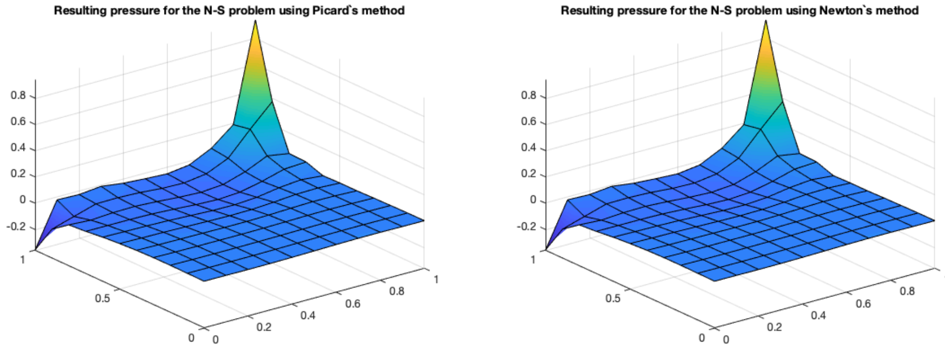

Figure 2: Pressure results for both methods for a 10 by 10 uniform mesh.

Then $\frac{\partial \mathbf{C}(\mathbf{v})\mathbf{v}}{\mathbf{v}} = \mathbf{C}(\mathbf{v}) + (\nabla \cdot \mathbf{v})\mathbf{Id}$, so the Jacobian can be written as

$$J = \begin{pmatrix} \mathbf{K} + \mathbf{C}(\mathbf{V}) + \mathbf{DVId} & \mathbf{G}^T \\ \mathbf{G} & 0 \end{pmatrix} \tag{7}$$

As we can see in Fig. 2, both methods converge to the same solution, that has a smoother but very similar shape than the Stokes solution.