Problem description

This first part of the report is aimed at analyzing the methods used to solve a Stokes problem. The Stokes equation is particular form of the general equation of motion describing a fluid where the inertial forces are small compared to the viscous forces, ie. low Reynolds's number. This entails the absence of the inertial term. An additional constraint in the Stokes problem is the assumption of an incompressibility in the fluid. Solving such problem using finite elements provide some difficulties. This is due to the fact that pressure is acting as a Lagrange multiplier. This could be handled by having a proper combination of the interpolation spaces, namely velocity and pressure. Depending on the type of element used and its compliance with the LBB condition, instabilities might occur. This could be handled using stabilization techniques such as SUPG and GLS similar to steady convection diffusion problems.

The problem in hand is described as a one*one square domain. The boundary conditions (BC) are as follows: three sides are fixed, while the upper side has a prescribed velocity of magnitude one in the x-direction. The pressure is set to zero at the lower left corner. The described BCs induce a discontinuity at the upper two corners. This results in a singularity in the pressure solution.

The problem is governed by the following differential equation:

$$\begin{aligned} -\nu \nabla^2 \boldsymbol{v} + \nabla p &= \boldsymbol{b} \quad && \text{in } \Omega \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} &= 0 \quad && \text{in } \Omega \end{aligned}$$

Equation 1 Governing differential equation

The Galerkin discretization leads to the following matrix form:

$$\begin{pmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{h} \end{pmatrix}$$

Equation 2 Matrix form of the system to be solved

Where:

$$\begin{aligned} \mathbf{K} &\leftarrow \int_\Omega [\text{grad } \mathbf{N}]^T [\text{grad } \mathbf{N}] d\Omega \\ \mathbf{G} &\leftarrow -\int_\Omega \widehat{\mathbf{N}}^T \mathbf{D} d\Omega \\ \mathbf{f} &\leftarrow \int_\Omega \mathbf{N}^T \boldsymbol{f} d\Omega \end{aligned}$$

In this analysis, four type of elements are to be used: bilinear shape function for pressure and velocity using a triangular element (P1P1) and a quadrilateral element (Q1Q1); and a biquadratic shape function for the velocity and a bilinear shape function for the pressure using a triangular element (P2P1) and a quadrilateral element (Q2Q1). However, P1P1 and Q1Q1 elements don't satisfy the LBB condition thus result in an unstable solution. This requires the application of a stabilization technique namely the GLS method. This leads to the following matrix form after a Galerkin discretization for linear elements:

$$\begin{pmatrix} \mathbf{K} & \mathbf{G}^T \\ -\mathbf{G} & \overline{\mathbf{L}} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \overline{\mathbf{f}}_q \end{pmatrix}$$

Equation 3 Matrix form of the system to be solved with stabilization

Where:

$$\overline{\mathbf{L}} \leftarrow \sum_e \int_{\Omega_e} \tau_1 (\boldsymbol{\nabla} q) \cdot (\boldsymbol{\nabla} p) d\Omega$$

$$\overline{\mathbf{f}}_q \quad \leftarrow \quad \sum_e \int_{\Omega_e} \tau_1 (\boldsymbol{\nabla} q) \cdot (-\boldsymbol{f}) d\Omega$$

$$\tau_1 = \alpha_0 \frac{h^2}{4\nu}$$

$$\alpha_0 = 1/3$$

Results

Figure 1 and Figure 2 below show the results for pressure using P2P1 and Q2Q1 elements. It could be seen that the solution is smooth, and no instabilities are present.

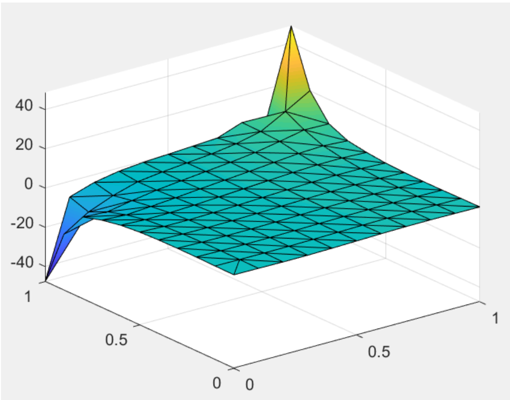

Figure 1 Pressure solution using P2P1 element

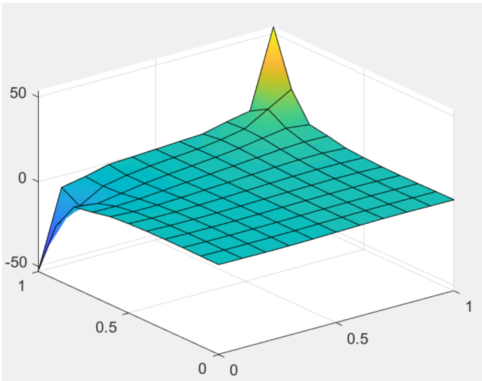

Figure 2 Pressure solution using Q2Q1 element

Figure 3 show the result for pressure using the P1P1 element. It could be seen that the solution obtained is not stable thus requiring the application pf a stabilization technique. Figure 4 show the result for pressure using the P1P1 element while applying the GLS stabilization method.

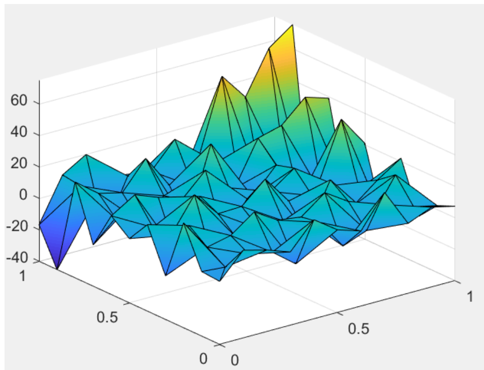

Figure 3 Unstable pressure solution using P1P1 element

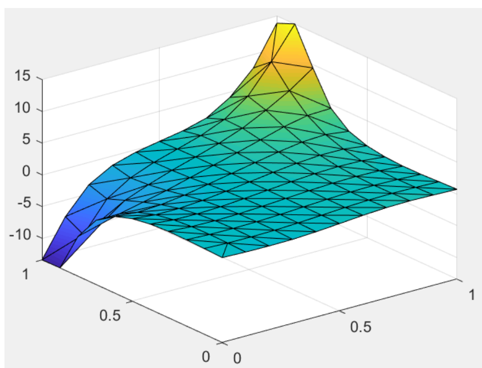

Figure 4 Stable pressure solution using P1P1 element

Figure 5 Figure 3show the result for pressure using the P1P1 element. It could be seen that the solution obtained is not stable thus requiring the application pf a stabilization technique. Figure 4Figure 6 show the result for pressure using the P1P1 element while applying the GLS stabilization method.

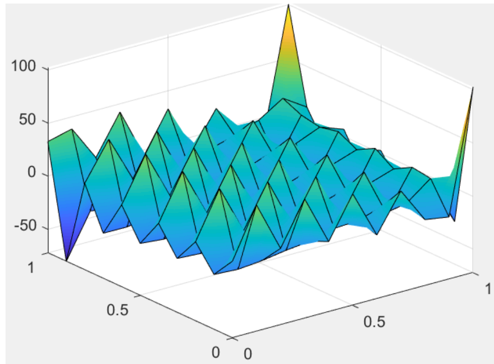

Figure 5 Unstable pressure solution using Q1Q1 element

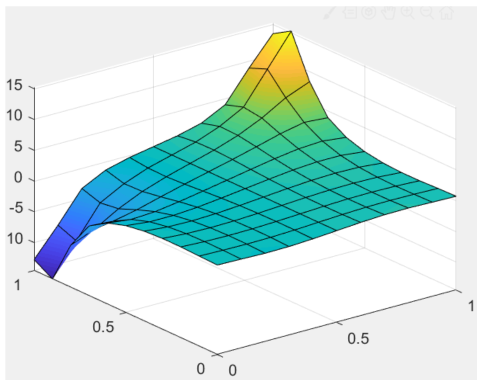

Figure 6 Stable pressure solution using Q1Q1 element

It could be concluded that the incompressibility condition assumed in the Stokes problem induce instabilities in the solution if the LBB condition is not satisfied. In order to reduce instabilities, stabilization techniques could be applied to the system. The effect of the application of such stabilization methods such as the GLS method is shown above.

Code

Modifications to the EleMatStokes function in the StokesSystem function:

```
Ke = Ke + (Nx'*Nx+Ny'*Ny)*dvolu;
Ge = Ge - NP_ig'*dN*dvolu;
x_ig = N_ig(1:ngeom)*Xe;
f_igaus = SourceTerm(x_ig)
fe = fe + Ngp'*f_igaus*dvolu;

%Stabilization for linear elements
```

```
h = 0.1;
tau1 = (1/3) * (h^2/(4*mu));
Le = Le + tau1 * (nxP'*nxP+nyP'*nyP) * dvoluP;

fqe = 1;
fqe = fqe - tau1 * Nx*f_igaus*dvolu
```

<u>Modifications to the StokesSystem function:</u>

```
L(TPe_dof,TPe_dof) = L(TPe_dof,TPe_dof) + Le;
```

<u>Modifications to the mainStokes script:</u>

```
Lred = L ;
Lred(1,:) = [];
Lred(:,1) = [];


if degreeV == 1
    LL = Lred;
else
    LL = zeros(nunkP);
end

A = [Kred   Gred';
     -Gred   LL ];
b = [fred; fqred];
```