Assignement 4

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Finite Elements in Fluids

16 de junio de 2019

1. Propagation of a steep front

The unsteady-convection equation governing the propagation of a steep front is the following:

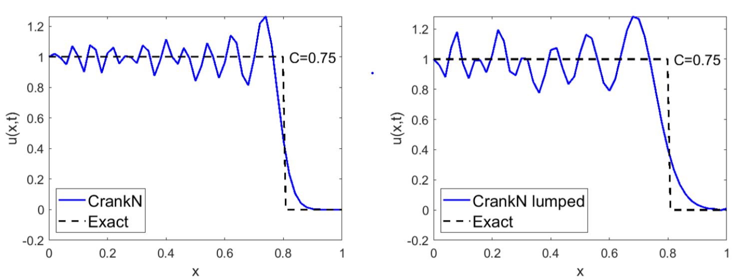
$$\begin{cases} u_t + au_x = 0 \quad , x \in (0,1) \ , t \in (0, 0{,}6] \\ u(x,0) = u_0(x) \quad , x \in (0,1) \\ u(0,t) = 1 \ \ t \in (0, 0{,}6] \end{cases}$$

$$\mathrm{u}_0(x) = \begin{cases} 1 \ if \ x \ <= \ 0{,}2, \\ 0 \ otherwise \end{cases}$$

The convection velocity takes the value a = 1 and the space and time discretization are given by $\Delta x = 0{,}02$ *and* $\Delta t = 0{,}015$. *Thus, the Courant number can be calculated as described.*

$$C = \frac{a|\Delta t|}{\Delta x} = 0{,}75 \tag{1}$$

1.1. Solve the problem using the Crank-Nicholson scheme in time and linear nite element for the Galerkin scheme in space. Is the solution accurate?

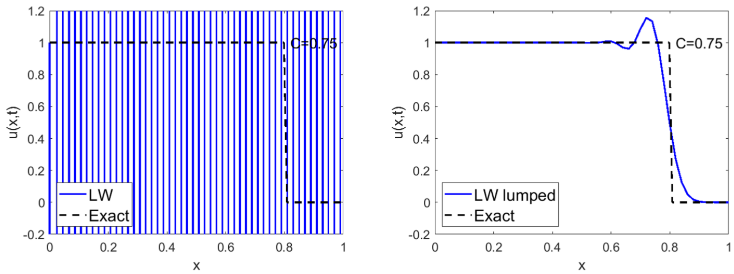


(a) Consistent mass matrix (b) Lumped mass matrix

Figura 1: Crank-Nicholson scheme

As we can see, the results are more accurate when a consistent mass matrix is used, as shown in gure (a), and there is a bigger error if a lumped mass matrix is used, as shown in gure (b). Nevertheless, both solutions are stable.

1.2. Solve the problem using the second-order Lax-Wendroff method. Can we expect the solution to be accurate? If not, what changes are necessary? Comment the results.



(a) Consistent mass matrix (b) Lumped mass matrix

Figura 2: Lax-Wendroff scheme

As expected, the Lax-Wendroff method is severely unstable for the Courant number considered ($C^2 > 0{,}33$), *ending with useless results.*
However, using the lumped mass matrix was effective in avoiding oscillations and provided an acceptable result.

1.3. Solve the problem using the third-order explicit Taylor-Galerkin method. Comment the results.

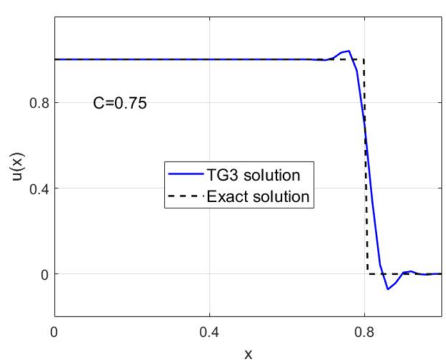


Figura 3: TG3 scheme

This method has a third order precision in time and therefore the error is lower in comparison with Lax-Wendroff and Crank-Nicholson. Moreover, $C^2 < 1$, *the scheme is stable.*

2. Burger's equation

The Burger's Equation can be solved via three different schemes, namely the Explicit, the Implicit Picard's method and the implicit Newton-Raphson's method. The proposed problem is defined on the [0,4] domain and it's initial condition is depicted on Figure 4.

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

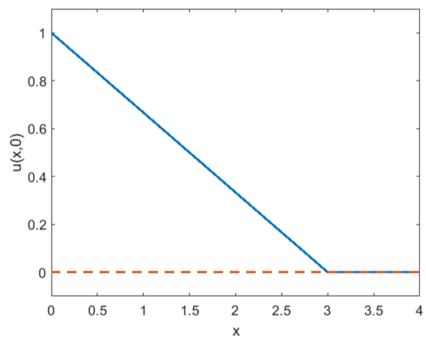


Figura 4: Initial Condition

Such initial conditions generate discontinuous solutions, requiring the usage of the vanishing viscosity approach. A viscosity is added to Equation the previous equations as seen on Equation (2).

$$u_t + tt_x = \epsilon u_{xx} \tag{2}$$

The result is, then, acquired as the viscosity tends to zero. After the Galerkin discretization the Burger's equation can be written as:

$$M\frac{\Delta U}{\Delta t} + C(U)U + \epsilon KU = 0 \tag{3}$$

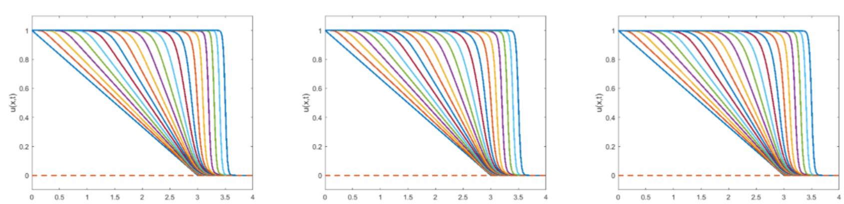
The Newton-Raphson method consists of solving the equation f(Un+1) = 0 every time step, where f(U) is giving by:

$$f(U) = (M + \Delta t C(U) + \epsilon \Delta t K)U - MU^n \tag{4}$$

Then, an iteration is made starting from the previous time-step Un+1 = Un until it converges according to a specified tolerance using the following expression:

$$U_{k+1}^{n+1} = U_k^{n+1} - J^{-1}(U_k^{n+1})f(U_k^{n+1}) \tag{5}$$

The results for all methods are presented on Figure 5, followed by a comparison of all methods on the last time-step on Figure 6.



(a) Explicit (b) Implicit Picard's (c) Implicit NR's

Figura 5: Solution for each scheme

As we can see all methods perform similarly for the given conditions of discretization, time-step and tolerance. However, the Newton-Raphson's method has a quadratic convergence as opposed to a linear convergence of the Picard's method, which might play a significant role on computational cost when requiring the same accuracy.

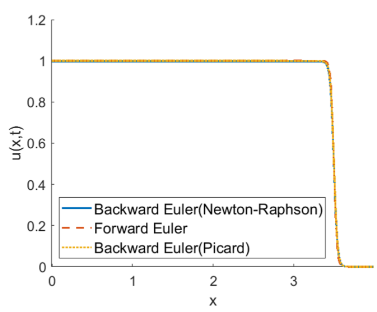


Figura 6: Comparison at final time-step