FEF-HW4: Numerical examples on Unsteady convective transport models and Compressible flows

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1 Unsteady convective transport models

Considering the one dimensional transient convection equation:

$$u_t + au_x = 0 \text{ on } x \in \Omega = (0,3) \text{ and } t \in (0, t_{End}) \tag{1}$$

with Dirichlet Boundary conditions $u(t,0) = u(t,3) = 0\ \forall t$ and Initial condition $u(0,x) = \frac{1+cos(\pi(x-x_0)/\sigma)}{2}$ if $|x - x_0| \leq \sigma$ and $u(0,x) = 0$ otherwise.

Where the parameter values are $a = 1$ and $\sigma = 0.12$.

1.1 Time discretization

At each method we will consider a different time discretization schemes. As notation we will use $u^n = u(t^n, x) = u(n \cdot \Delta t, x)$ and $\Delta u^n = u^{n+1} - u^n$.

- **Crank-Nicholson:** For the Crank-Nicholson method we will consider the following scheme

$$\frac{\Delta u^n}{\Delta t} + \frac{a \Delta u_x^n}{2} = -a u_x^n \tag{2}$$

Weak form: Consider $\omega \in H^1(\Omega)$ such that $\omega(0) = \omega(3) = 0$

$$\frac{1}{\Delta t}\int_\Omega \omega \Delta u^n d\Omega + \frac{a}{2}\int_\Omega \omega \Delta u_x^n d\Omega = -a \int_\Omega \omega u_x^n d\Omega \tag{3}$$

- **Lax-Wendroff:** For the Lax-Wendroff method we will consider the following scheme

$$\frac{\Delta u^n}{\Delta t} = -a u_x^n + \frac{\Delta t}{2} a^2 u_{xx}^n \tag{4}$$

Weak form: Consider $\omega \in H^1(\Omega)$ such that $\omega(0) = \omega(3) = 0$

$$\frac{1}{\Delta t}\int_\Omega \omega \frac{\Delta u^n}{\Delta t} d\Omega = -a \int_\Omega \omega u_x^n d\Omega + \frac{\Delta t}{2} a^2 \int_\Omega \omega u_{xx}^n d\Omega \tag{5}$$

Integrating by parts the last term in on the RHS we get eq. 6 since we had imposed $\omega(0) = \omega(3) = 0$.

$$\frac{1}{\Delta t}\int_\Omega \omega \frac{\Delta u^n}{\Delta t} d\Omega = -a\int_\Omega \omega u_x^n d\Omega - \frac{\Delta t}{2}a^2 \int_\Omega \omega_x u_x^n d\Omega \tag{6}$$

- **Third order Taylor-Galerkin:** For the Third order Taylor-Galerkin method we will consider the following scheme

$$\frac{\Delta u^n}{\Delta t} - \frac{\Delta t}{6}\Delta a^2 u_{xx}^n = -a u_x^n + \frac{\Delta t}{2}a^2 u_{xx}^n \tag{7}$$

Weak form: Consider $\omega \in H^1(\Omega)$ such that $\omega(0) = \omega(3) = 0$

$$\frac{1}{\Delta t}\int_\Omega \omega \Delta u^n d\Omega - \frac{\Delta t}{6}a^2 \int_\Omega \omega \Delta u_{xx}^n d\Omega = -a\int_\Omega \omega u_x^n d\Omega + \frac{\Delta t}{2}a^2 \int_\Omega \omega u_{xx}^n d\Omega \tag{8}$$

Integrating by parts the last term in on the RHS and the last one on the LHS we get eq. 9 since we had imposed $\omega(0) = \omega(3) = 0$.

$$\frac{1}{\Delta t}\int_\Omega \omega \Delta u^n d\Omega + \frac{\Delta t}{6}a^2 \int_\Omega \omega_x \Delta u_x^n d\Omega = -a\int_\Omega \omega u_x^n d\Omega - \frac{\Delta t}{2}a^2 \int_\Omega \omega_x u_x^n d\Omega \tag{9}$$

1.2 Space discretization

Now we consider a FEM discretization in space $u(x,t) = \sum_{i=1}^{n_{Nodes}} u_j(t)N_j(x)$ where the nodal values deppend on the time and the basis functions of $\{H^1(\Omega) \cap \{u(0) = u(3) = 0\}\}$ deppend on the space. Considering $\omega = N_i(x)$ and substituting in every weak form we have:

- **Crank-Nicholson:**

$$A_{CN}\mathbf{\Delta u^n} = B_{CN}\mathbf{u^n} \Rightarrow \left[\frac{1}{\Delta t}M + \frac{a}{2}C\right]\mathbf{\Delta u^n} = \left[-aC\right]\mathbf{u^n} \tag{10}$$

- **Lax-Wendroff:**

$$A_{LW}\mathbf{\Delta u^n} = B_{LW}\mathbf{u^n} \Rightarrow \left[\frac{1}{\Delta t}M\right]\mathbf{\Delta u^n} = \left[-aC - \frac{\Delta t}{2}a^2 K\right]\mathbf{u^n} \tag{11}$$

- **Third order Taylor-Galerkin:**

$$A_{TG3}\mathbf{\Delta u^n} = B_{TG3}\mathbf{u^n} \Rightarrow \left[\frac{1}{\Delta t}M + \frac{\Delta t}{6}a^2 K\right]\mathbf{\Delta u^n} = \left[-aC - \frac{\Delta t}{2}a^2 K\right]\mathbf{u^n} \tag{12}$$

where $M_{ij} = \int_\Omega N_i N_j d\Omega$, $K_{ij} = \int_\Omega N_{xi} N_{xj} d\Omega$ and $C_{ij} = \int_\Omega N_i N_{xj} d\Omega$

1.3 Courant Number and results

We know the stability conditions of our methods are

	CN	LW	TG3
Type of method	Implicit	Explicit	Explicit
Order of accuracy	2nd	2nd	3rd
Stability condition	Unconditional	$Cu^2 \leq 1/3$	$Cu \leq 1$

Table 1: Stability conditions for the 3 methods.

Where Cu represents the Courant number that is $Cu = |a|\Delta t/\Delta x$.

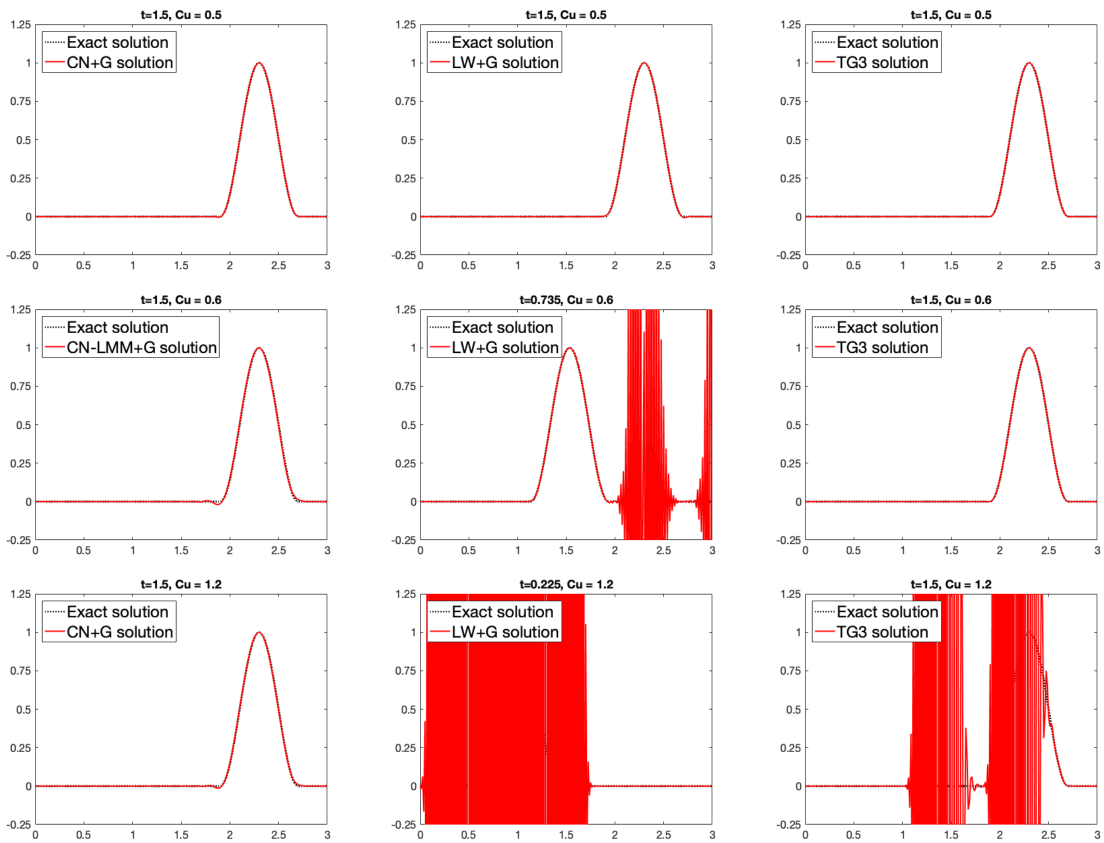

Figure 1: Results for different times, $Cu = 0.5, 0.6, 1.2$ and the 3 methods.

As we can see in Figure 1, changing the number of time steps and as a consequence Δt, we can see different behaviours. For $Cu = 0.5 \Rightarrow Cu^2 = 0.25 \leq 1/3$, the 3 methods are stable and we can see a more accurate solution for the 3rd order one (TG3). For $Cu = 0.6 \Rightarrow Cu^2 = 0.36 \in [1/3, 1]$, the Law-Wendroff method is unstable and the other ones are not. Finally, for $Cu = 1.2 \Rightarrow Cu^2 = 1.44 > 1$ the third order Taylor-Galerkin method is also unstable but the Crank-Nicholson is not. So we confirm the stability conditions from table **??**.

2 Compressible flows

We consider the Burgers' perturbed equations

$$u_t + uu_x = \epsilon u_{xx} \tag{13}$$

$$u(x,0) = u_0(x) \tag{14}$$

So the weak form is: Consider $\omega \in \{H^1(\Omega) \cap \omega(x) = 0, \forall x \in \delta\Omega\}$

$$\int_\Omega \omega u_t d\Omega + \int_\Omega \omega u u_x d\Omega = \epsilon \int_\Omega \omega u_{xx} d\Omega \tag{15}$$

Integrating by parts the RHS (taking into account that $\omega(x) = 0, \forall x \in \delta\Omega$),

$$\int_\Omega \omega u_{xx} d\Omega = -\int_\Omega \omega_x u_x d\Omega + \int_{\delta\Omega} \omega u_x \cdot \mathbf{n} d(\delta\Omega) = -\int_\Omega \omega_x u_x d\Omega \tag{16}$$

We end with the expression:

$$\int_\Omega \omega u_t d\Omega + \int_\Omega \omega u u_x d\Omega + \epsilon \int_\Omega \omega_x u_x d\Omega \tag{17}$$

Considering now a FEM discretization of the space $u(x,t) \approx u^h(x,t) = \sum_{j=1}^{n_{Nodes}} u_j(t) N_j(x)$, and we can consider $\omega = N_i(x)$ for $i = 1, ..., n_{Nodes}$. Applying it to the weak form (eq. 17), we end with the non-linear system of ODE's

$$\mathbf{M}\dot{\mathbf{U}} + \mathbf{C}(\mathbf{U})\mathbf{U} + \epsilon \mathbf{K}\mathbf{U} = \mathbf{0} \tag{18}$$

where $M_{ij} = \int_\Omega N_i(x) N_j(x) d\Omega$, $K_{ij} = \int_\Omega (N_i)_x (N_j)_x d\Omega$ and $(C(U))_{ij} = \int_\Omega N_i(\sum_k \mathbf{U}_k N_k)(N_j)_x$.

If we apply a Backward-Euler scheme to the time derivative we have:

$$\mathbf{M}\frac{\mathbf{U}^{n+1} - \mathbf{U}^n}{\Delta t} + \mathbf{C}(\mathbf{U}^{n+1})\mathbf{U}^{n+1} + \epsilon \mathbf{K}\mathbf{U}^{n+1} = \mathbf{0} \Rightarrow (\mathbf{M} + \Delta t(\mathbf{C}(\mathbf{U}^{n+1}) + \epsilon \mathbf{K}))\mathbf{U}^{n+1} = \mathbf{M}\mathbf{U}^n \tag{19}$$

that is a non-linear system.

To solve this system we will use two methods: Picard Method and Newton-Raphson method.

2.1 Picard method

At each time step we have to solve $\mathbf{A}(\mathbf{U}^{n+1})\mathbf{U}^{n+1} = (\mathbf{M} + \Delta t(\mathbf{C}(\mathbf{U}^{n+1}) + \epsilon \mathbf{K}))\mathbf{U}^{n+1} = \mathbf{M}\mathbf{U}^n$

Our method will follow the scheme:

- Initial value: ${}^0\mathbf{U}^{n+1} = \mathbf{U}^n$.
- Until $||{}^{k+1}\mathbf{U}^{n+1} -{}^k \mathbf{U}^{n+1}|| > tol$: ${}^{k+1}\mathbf{U}^{n+1} = \mathbf{A}^{-1}({}^k\mathbf{U}^{n+1})(\mathbf{M}\mathbf{U}^n)$

2.2 Newton-Raphson method

At each time step we have to solve $\mathbf{f}(\mathbf{U}^{n+1}) = 0$ with $\mathbf{f}(\mathbf{U}) = (\mathbf{M} + \Delta t(\mathbf{C}(\mathbf{U}) + \epsilon\mathbf{K}))\mathbf{U} - \mathbf{M}\mathbf{U}^n$

Our method will follow the scheme:

- Initial value: ${}^{0}\mathbf{U}^{n+1} = \mathbf{U}^n$.
- Until $||\Delta^{k+1}\mathbf{U}^{n+1}|| > tol$:

$$\Delta^{k+1}\mathbf{U}^{n+1} = -\mathbf{J}^{-1}({}^{k}\mathbf{U}^{n+1})\mathbf{f}({}^{k}\mathbf{U}^{n+1}) \tag{20}$$

$${}^{k+1}\mathbf{U}^{n+1} =^{k} \mathbf{U}^{n+1} + \Delta^{k+1}\mathbf{U}^{n+1} \tag{21}$$

where $\mathbf{J} = \frac{d\mathbf{f}}{\mathbf{U}} = (\mathbf{M} + \Delta t(\mathbf{C}(\mathbf{U}) + \epsilon\mathbf{K})) + \frac{d\mathbf{C}}{\mathbf{U}}\mathbf{U}$.

So we have to define $\frac{d\mathbf{C}}{\mathbf{U}} = C'$, where $C'_{ijk}(\mathbf{U}) = \frac{d\mathbf{C}_{ij}}{\mathbf{U}_k}(\mathbf{U}) = \int_\Omega N_i N_k (N_j)_x$ that is independent of $\mathbf{U}$. Thus $\frac{d\mathbf{C}}{\mathbf{U}}\mathbf{U} = \sum_k^{n_{Nodes}} C'_{ijk}\mathbf{U}_k$.

I can not plot the results of the Newton-Raphson method, since I do not know how to compute C'.