Rotating cone problem:

The given problem is a transient 2D homogeneous convection problem considers the convection of a product-cosine hill in a pure rotation velocity field with the initial condition given as:

$$u(\mathbf{x},0) = \begin{cases} \frac{1}{4}(1+\cos\pi X_1)(1+\cos\pi X_2) & {X_1}^2+{X_2}^2 \leq 1 \\ 0 & Otherwise \end{cases}$$

where $\mathbf{X} = (\mathbf{x}-\mathbf{x}_0)/\sigma$, and the boundary condition is $u = 0 \ on \ \Gamma^{in}$. The initial position of the center and the radius of the cosine hill are $\mathbf{x}_0$ and $\mathbf{a}$, respectively. In the examples they are chosen as $\mathbf{x}_0 = (\frac{1}{6}, \frac{1}{6}) \ and \ \mathbf{a} = 0.2$. The convection field is a pure rotation one with unit angular velocity, namely $\mathbf{a}(\mathbf{x}) = (-x_2, x_1)$. A uniform mesh of $30x30$ four-node elements over the unit square $[-\frac{1}{2}, \frac{1}{2}] * [-\frac{1}{2}, \frac{1}{2}]$is employed in the calculations.

The numerical solutions have been computed after a full revolution completed in 200 time steps ($\Delta t = 2\pi/200$). They have been computed using the following finite element schemes in the Galerkin formulation, namely:

- Lax-Wendroff + Galerkin (and with lumped mass matrix);
- Crank-Nicolson +Galerkin (and lumped mass matrix);
- The third-order explicit Taylor-Galerkin scheme (TG3);
- A third order two-step TG3-2S method;
- Two-step fourth-order method

Lax-Wendroff method:

The Lax-Wendroff Scheme is:

$\frac{\Delta u}{\Delta t} = -\mathbf{a}.\boldsymbol{\nabla} u^n + \frac{\Delta t}{2}(\mathbf{a}.\boldsymbol{\nabla})^2 u^n + s^n + \frac{\Delta t}{2}({s_t}^n - \mathbf{a}.\boldsymbol{\nabla} s^n)$

The weak form for the Lax-Wendroff method can be written as considering $s = 0$ and $h = 0$:

$(w, \frac{\Delta u}{\Delta t}) = (\mathbf{a}.\boldsymbol{\nabla} w, u^n - \frac{\Delta t}{2}(\mathbf{a}.\boldsymbol{\nabla})u^n) - ((\mathbf{a}.\mathbf{n})w, u^n - \frac{\Delta t}{2}(\mathbf{a}.\boldsymbol{\nabla})u^n)_{\Gamma^{out}}$
In the supplied code : A = M represents the term $(w, \Delta u)$;

In, B $= dt * (C - (dt/2) * K - Mo + (dt/2) * Co); C \ represents \ the \ term \ (\mathbf{a}.\boldsymbol{\nabla} w, u^n)$

$K \ represents \ the \ term \ (\mathbf{a}.\boldsymbol{\nabla} w)(\mathbf{a}.\boldsymbol{\nabla} u^n)$

$Mo \ represents \ ((\mathbf{a}.\mathbf{n})w, u^n)_{\Gamma^{out}}$

Co represents $((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\boldsymbol{\nabla})u^n)_{\Gamma^{out}}$

As s=0; it is not required to describe the terms in $f = dt * (v1 + (dt/2) * (v2 - vo))$;

But, v1 represents (w, s^n);
v2 represents $(\mathbf{a}.\boldsymbol{\nabla} w, s^n)$
v0 represents $((\mathbf{a}.\mathbf{n})w, s^n)_{\Gamma^{out}}$

In the the second-order Lax-Wendroff method combined with bilinear elements and a diagonal mass representation, in A = Md, Md represents the diagonal mass representation of M.
In B $= dt*(C-(dt/2)*K-Mod+(dt/2)*Co)$, Mod represents the diagonal mass representation of Mo.

TG3 method:

The TG3 Scheme is with s=0:

$(1-\frac{\Delta t^2}{6}(\mathbf{a}.\boldsymbol{\nabla})^2)\frac{\Delta u}{\Delta t} = -\mathbf{a}.\boldsymbol{\nabla}u^n + \frac{\Delta t}{2}(\mathbf{a}.\boldsymbol{\nabla})^2u^n$

The weak form for the TG3 method can be written as considering $s=0$ and $h=0$:

$(w, \frac{\Delta u}{\Delta t}) + \frac{\Delta t^2}{6}(\mathbf{a}.\boldsymbol{\nabla}w, (\mathbf{a}.\boldsymbol{\nabla}\frac{\Delta u}{\Delta t}) - \frac{\Delta t^2}{6}((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\boldsymbol{\nabla}\frac{\Delta u}{\Delta t}))_{\Gamma^{out}}$

$= (\mathbf{a}.\boldsymbol{\nabla}w, u^n - \frac{\Delta t}{2}(\mathbf{a}.\boldsymbol{\nabla})u^n) - ((\mathbf{a}.\mathbf{n})w, u^n - \frac{\Delta t}{2}(\mathbf{a}.\boldsymbol{\nabla})u^n)_{\Gamma^{out}}$

In the supplied code :

In $A = M+(dt^2/6)*(K-Co)$: M represents the term $(w, \Delta u)$,K *represents the term* $(\mathbf{a}.\boldsymbol{\nabla}w)(\mathbf{a}.\boldsymbol{\nabla}\Delta u)$and Co represents $((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\boldsymbol{\nabla})\Delta u)_{\Gamma^{out}}$;

In, B $= dt*(C-(dt/2)*K-Mo+(dt/2)*Co)$: C *represents the term* $(\mathbf{a}.\boldsymbol{\nabla}w, u^n)$

K *represents the term* $(\mathbf{a}.\boldsymbol{\nabla}w)(\mathbf{a}.\boldsymbol{\nabla}u^n)$

Mo *represents* $((\mathbf{a}.\mathbf{n})w, u^n)_{\Gamma^{out}}$

Co represents $((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\boldsymbol{\nabla})u^n)_{\Gamma^{out}}$

As s=0; it is not required to describe the terms in $f = dt*((dt/2)*(v2-vo)+v1)$;

To compare the accuracy of the various observed explicit methods, the maximum and minimum values of the computed solutions are provided in Figures 1,2 and 3. The greater accuracy of the finite element schemes employing a consistent (TG2) or generalized (TG3) mass matrix is clearly apparent. Admittedly, the consistent finite element schemes are computationally more expensive than the Lax-Wendroff scheme using a diagonal mass matrix, because the solution of a banded (symmetric) linear system is required at each time step.

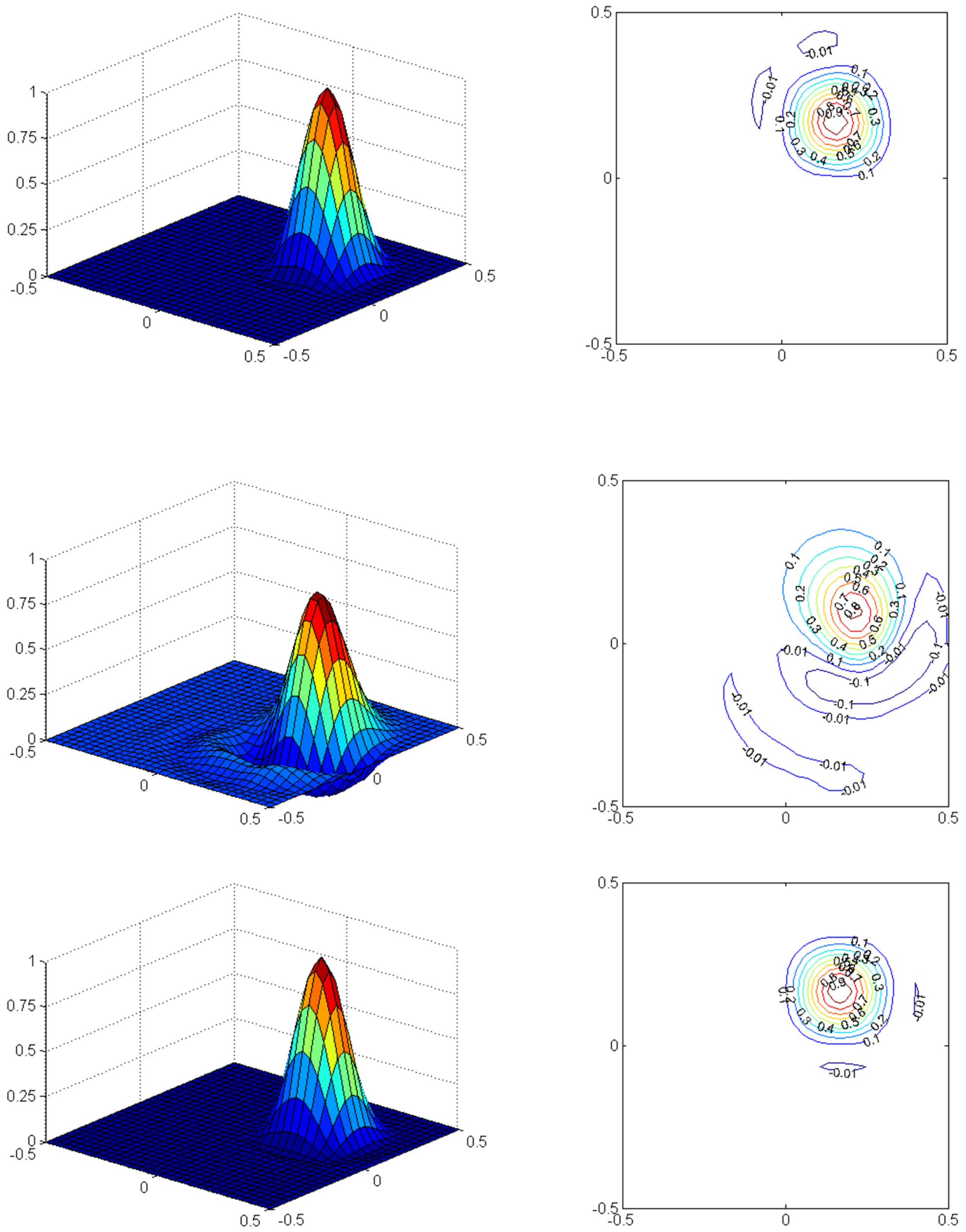

Figure 1: Convection of a cosine hill in a pure rotation velocity field: comparison of the numerical solutions after a complete revolution calculated with $\Delta t = 2\pi/200$ by means of (Top): Lax-Wendroff/diagonal mass scheme $(u_{max} = 0.8186, u_{min} = -0.1774); (Middle) : TG2(u_{max} = 0.9830, u_{min} = -0.0186); and\ (Bottom) : TG3(u_{max} = 0.9835, u_{min} = -0.0148)$.

Crank-Nicolson Scheme:

The CN Scheme is:

$\frac{\Delta u}{\Delta t} + \frac{1}{2}(\mathbf{a}.\boldsymbol{\nabla})\Delta u = -\mathbf{a}.\boldsymbol{\nabla} u^n$

The weak form for the CN method can be written as considering $s = 0$ and $h = 0$:

$(w, \frac{\Delta u}{\Delta t}) - \frac{1}{2}(\boldsymbol{\nabla} w, \mathbf{a}\Delta u) + \frac{1}{2}((\mathbf{a}.\mathbf{n})w, \Delta u)_{\Gamma^{out}} = (\boldsymbol{\nabla} w, \mathbf{a}u^n) - ((\mathbf{a}.\mathbf{n})w, u^n)_{\Gamma^{out}}$

In the supplied code for the Galerkin formulation + Crank-Nicolson with consistent mass matrix:

In $A = M-(dt/2)*C+(dt/2)*Mo$: M represents the term $(w, \Delta u)$,$C\ represents\ the\ term$ $(\boldsymbol{\nabla} w, \mathbf{a}\Delta u)$and Mo represents $((\mathbf{a}.\mathbf{n})w, \Delta u)_{\Gamma^{out}}$;

In, B $= dt * C - dt * Mo : C\ represents\ the\ term\ (\mathbf{a}.\boldsymbol{\nabla} w, u^n)$

$Mo\ represents\ ((\mathbf{a}.\mathbf{n})w, u^n)_{\Gamma^{out}}$

Co represents $((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\boldsymbol{\nabla})u^n)_{\Gamma^{out}}$

As s=0; it is not required to describe the terms in $f = dt * v1$;

In the supplied code for the Galerkin formulation + Crank-Nicolson with lumped mass matrix: Md represents the diagonal mass representation of M and Mod represents the diagonal mass representation of Mo.

Three increasing values of the time step are employed, which correspond to a complete revolution of the cone in, respectively, 120, 60 and 30 time steps. In this way, we shall appraise the behavior of the Crank—Nicolson/Galerkin method beyond the stability limit of the explicit TG3 scheme. The numerical results after one complete revolution of the cone are displayed in Figure 2. The Crank—Nicolson scheme with the Galerkin formulation represents a non-dissipative method in pure convection. Moreover, the phase accuracy of the method decreases when the time step is increased. As a result, significant non-physical oscillations develop as soon as the time-step size exceeds the stability limit of the explicit schemes.

The CN scheme employing a consistent mass matrix gives greater accuracy than the CN scheme using a diagonal mass matrix,but the later case is less expensive because the solution of a banded (symmetric) linear system is required at each time step. It can be oberved from the Figure 3.

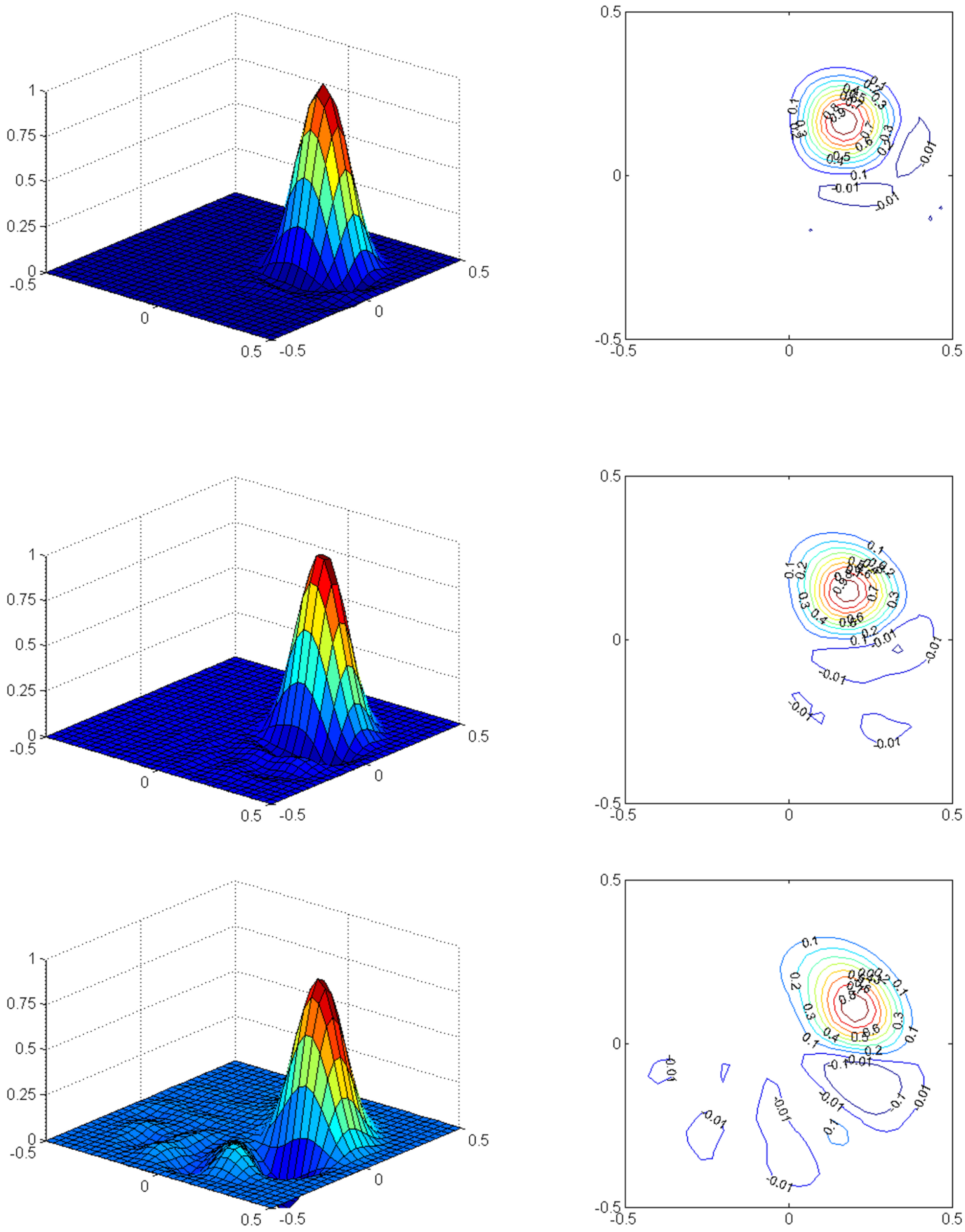

Figure 2: Convection of a cosine hill in a pure rotation velocity field using the Crank-Nicolson/Galerkin method: comparison of the numerical solutions after a complete revolution computed with calculated with by means of (Top): $\Delta t = 2\pi/120$ ($u_{max} = 0.9969, u_{min} = -0.0454$); (Middle):$\Delta t = 2\pi/60$ ($u_{max} = 0.9691, u_{min} = -0.1096$); and (Bottom): $\Delta t = 2\pi/30$ ($u_{max} = 0.8931, u_{min} = -0.2694$).

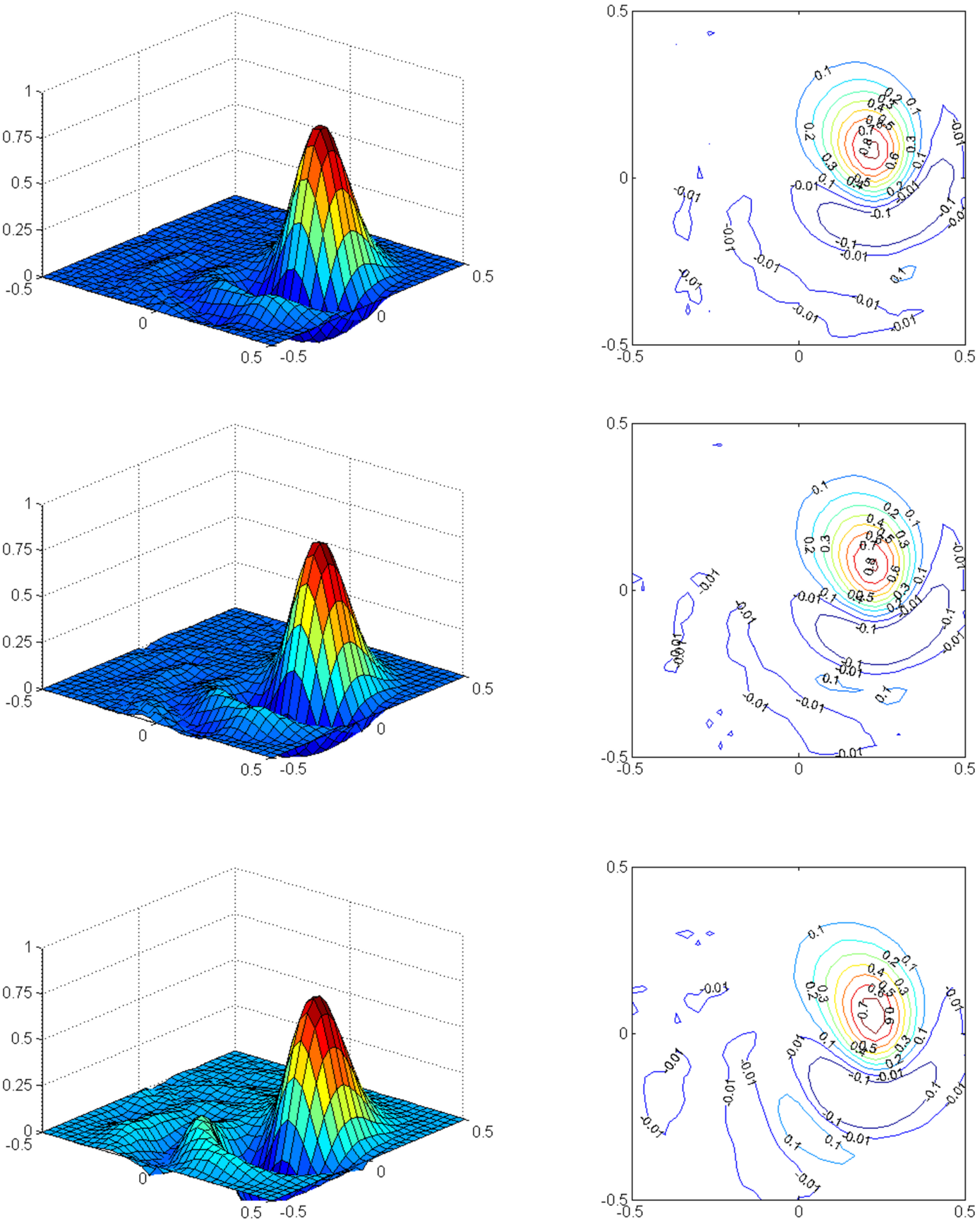

Figure 3: Convection of a cosine hill in a pure rotation velocity field using the Crank-Nicolson/Galerkin method with lumped mass matrix: comparison of the numerical solutions after a complete revolution computed with calculated with by means of (Top): $\Delta t = 2\pi/120$ ($u_{max} = 0.8216, u_{min} = -0.2149$); (Middle):$\Delta t = 2\pi/60$ ($u_{max} = 0.8147, u_{min} = -0.234$); and (Bottom): $\Delta t = 2\pi/30$ ($u_{max} = 0.7624, u_{min} = -0.3096$).

TG3-2S(Two-step methods):

The weak form of the TG3-2S scheme can be expressed as with $\alpha = 1/9$:

$$(w, \frac{\bar{u}^n - u^n}{\Delta t}) = \frac{1}{3}(\mathbf{a}.\boldsymbol{\nabla} w, u^n) - \alpha(\Delta t)(\mathbf{a}.\boldsymbol{\nabla} w, \mathbf{a}.\boldsymbol{\nabla} u^n) + \alpha(\Delta t)((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\boldsymbol{\nabla})u^n)_{\Gamma^{out}}$$

$$(w, \frac{u^{n+1} - u^n}{\Delta t}) = (\mathbf{a}.\boldsymbol{\nabla} w, u^n) - \frac{\Delta t}{2}(\mathbf{a}.\boldsymbol{\nabla} w, \mathbf{a}.\boldsymbol{\nabla} \bar{u}^n) + \frac{\Delta t}{2}((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\boldsymbol{\nabla})\bar{u}^n)_{\Gamma^{out}}$$

In the supplied code for the TG3-2S:

In $A1 = M$: M represents the term $(w, \bar{u}^n - u^n)$;

In, B1 $= -(dt/3) * C' - alpha * dt^2 * (K - Co) : -C'$ *represents the term* $(\mathbf{a}.\boldsymbol{\nabla} w, u^n)$

K *represents* $(\mathbf{a}.\boldsymbol{\nabla} w, \mathbf{a}.\boldsymbol{\nabla} u^n)$

Co represents $((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\boldsymbol{\nabla})u^n)_{\Gamma^{out}}$

In $A2 = M$: M represents the term $(w, u^{n+1} - u^n)$;

In, B2 $= -dt * C' : -C'$ *represents the term* $(\mathbf{a}.\boldsymbol{\nabla} w, u^n)$

In $C2 = -(dt^2/2) * (K - Co)$; K *represents* $(\mathbf{a}.\boldsymbol{\nabla} w, \mathbf{a}.\boldsymbol{\nabla} \bar{u}^n)$

Co represents $((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\boldsymbol{\nabla})\bar{u}^n)_{\Gamma^{out}}$

For the two-step fourth order method α is considered as $\frac{1}{12}$. It can be observed that both the TG3-2S and TG4-2S methods behave well in their stability limits. It can be observed from the Figure 4 and the Figure 5. TG4-2S is a fourth order method, where TG3-2S is a third order method.

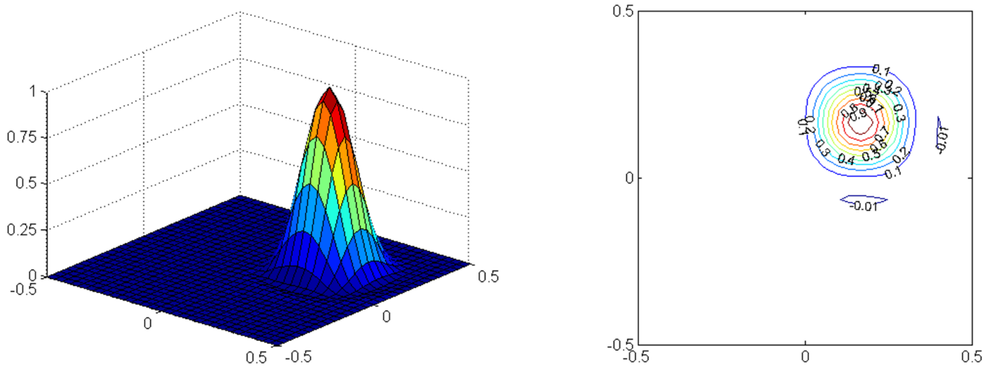

Figure 4: Convection of a cosine hill in a pure rotation velocity field: comparison of the numerical solutions after a complete revolution calculated with $\Delta t = 2\pi/200$ by means of TG3-2S ($u_{max} = 0.98284, u_{min} = -0.01494$)

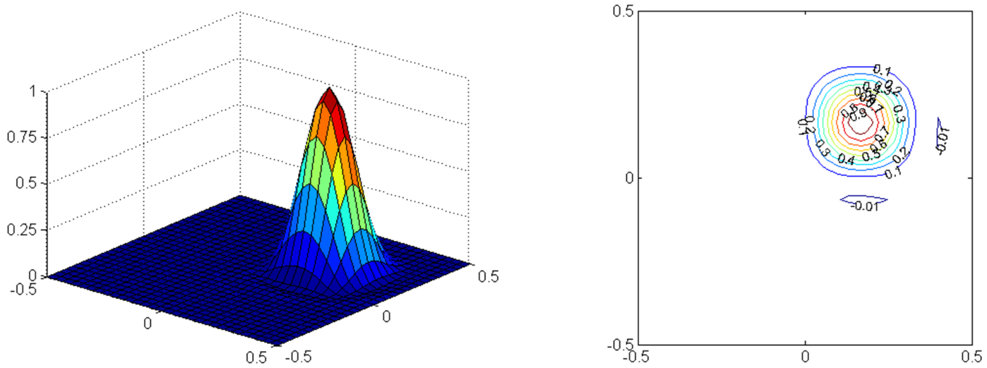

Figure 5: Convection of a cosine hill in a pure rotation velocity field: comparison of the numerical solutions after a complete revolution calculated with $\Delta t = 2\pi/200$ by means of TG4-2S ($u_{max} = 0.98284, u_{min} = -0.14939$)

CODE:

```
if meth == 1
    A = M;
    B = dt*(C - (dt/2)*K - Mo + (dt/2)*Co);
    f = dt*(v1 + (dt/2)*(v2-vo));
elseif meth == 2
    Md = diag(M*ones(numnp,1));
    Mod = diag(Mo*ones(numnp,1));
    A = Md;
    B = dt*(C - (dt/2)*K - Mod + (dt/2)*Co);
    f = dt*(v1 + (dt/2)*(v2-vo));
elseif meth == 3
    A = M +(dt^2/6)*(K - Co);
    B = dt*(C - (dt/2)*K - Mo + (dt/2)*Co);
    f = dt*((dt/2)*(v2 - vo) +v1);
elseif meth == 4
    A = M - (dt/2)*C + (dt/2)*Mo;
    B = dt*C - dt*Mo;
    f = dt*v1;
elseif meth == 5
    Md = diag(M*ones(numnp,1));
    Mod = diag(Mo*ones(numnp,1));
    A = Md - (dt/2)*C + (dt/2)*Mod;
    B = dt*C - dt*Mod;
    f = dt*v1;
elseif meth == 6
    A = M + (dt/2)*(C + C') + (dt^2/4)*K ;
    B = -dt*(C' + (dt/2)*K);
    f = dt*v1 + (dt^2/2)*v2;
elseif meth == 7            %TG3-2S
    alpha = 1/9;
    A1 = M;
    B1 = -(dt/3)*C'- alpha*dt^2*(K - Co);
    f1 = (dt/3)*v1 + alpha*dt^2*(v2 - vo);
    A2 = M;
    B2 = -dt*C';
    C2 = - (dt^2/2)*(K-Co);
    f2 = dt*v1 - (dt^2/2)*(v2 - vo);
elseif meth == 8              %TG4-2S
alpha = 1/12;
    A1 = M;
    B1 = -(dt/3)*C'- alpha*dt^2*(K - Co);
    f1 = (dt/3)*v1 + alpha*dt^2*(v2 - vo);
    A2 = M;
    B2 = -dt*C';
    C2 = - (dt^2/2)*(K-Co);
    f2 = dt*v1 - (dt^2/2)*(v2 - vo);
else
    error('Unavailable method')
end
```