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PERTURBED BURGERS' EQUATION

$$\begin{cases} u_t + u u_x = \varepsilon u_{xx} \\ u(x,0) = u_0(x) \longleftarrow \text{I.C.} \\ u(-1,t) = u(1,t) = 0 \longleftarrow \text{Dirichlet B.C.} \end{cases}$$

So we have: $f_x(u) = u u_x - \varepsilon u_{xx} \quad (u_t = -f_x(u))$

- $f(u) = \frac{u^2}{2} - \varepsilon u_x$
- $f_t(u) = u \cdot u_t - \varepsilon u_{xt} = u \cdot [-f_x(u)] - \varepsilon [u_t]_x = -u[f_x(u)] + \varepsilon [f_x(u)]_x$

$$= \varepsilon u \cdot u_{xx} - u^2 u_x + \varepsilon (u_x)^2 + \varepsilon u u_{xx} - \varepsilon^2 u_{xxx}$$

$$= 2\varepsilon u \cdot u_{xx} - u^2 u_x + \varepsilon (u_x)^2 - \varepsilon^2 u_{xxx}$$

ONE STEP TAYLOR - GALERKIN METHOD

$$u^{n+1} = u^n + \Delta t\, u_t^n + \frac{1}{2} (\Delta t)^2 u_{tt}^n$$

$$\begin{cases} u_t^n = -f_x(u^n) = -\left[\frac{(u^n)^2}{2} - \varepsilon u_x^n\right]_x \\ u_{tt}^n = -f_{xt}(u^n) = -f_{tx}(u^n) = \left[2\varepsilon u \cdot u_{xx} - u^2 u_x + \varepsilon (u_x)^2 - \varepsilon^2 u_{xxx}\right]_x \end{cases}$$

WEAK FORM: consider $w \in H^1([-1,1])$ such that $w(-1) = w(1) = w_x(-1) = w_x(1) = 0$

$$\int_{-1}^{1} u^{n+1} w\, dx = \int_{-1}^{1} u^n w\, dx - \Delta t \int_{-1}^{1} f_x(u^n) w\, dx - \frac{(\Delta t)^2}{2} \int_{-1}^{1} [f_t(u^n)]_x w\, dx$$

- Integrating by parts:

$$\to \int_{-1}^{1} f_x(u^n) w\, dx = \Big[f(u^n(1)) \overset{0\,(*)}{w(1)} - f(u^n(-1)) \overset{0\,(*)}{w(-1)}\Big] - \int_{-1}^{1} f(u^n) w_x\, dx$$

$$\to \int_{-1}^{1} [f_t(u^n)]_x w\, dx = \Big[f_t(u^n(1)) \overset{0\,(*)}{w(1)} - f_t(u^n(-1)) \overset{0\,(*)}{w(-1)}\Big] - \int_{-1}^{1} f_t(u^n) w_x\, dx$$

so we have,

$$\int_{-1}^{1} u^{n+1} w\, dx = \int_{-1}^{1} u^n w\, dx + \Delta t \int_{-1}^{1} f(u^n) w_x\, dx + \frac{(\Delta t)^2}{2} \int_{-1}^{1} f_t(u^n) w_x\, dx$$

$$\int_{-1}^{1} u^{n+1} w\, dx = \int_{-1}^{1} u^n w\, dx + \Delta t \int_{-1}^{1} \left[\frac{(u^n)^2}{2} - \varepsilon u_x^n + \frac{\Delta t}{2}\left(2\varepsilon u^n \cdot u_{xx}^n - (u^n)^2 u_x^n + \varepsilon (u_x^n)^2 - \varepsilon^2 u_{xxx}^n\right)\right] w_x\, dx$$

Also integrating by parts:

$$\to \int_{-1}^{1} u_{xxx}^n w_x\, dx = \Big[u_{xx}^n(1) \overset{0\,(*)}{w_x(1)} - u_{xx}^n(-1) \overset{0\,(*)}{w_x(-1)}\Big] - \int_{-1}^{1} u_{xx}^n w_{xx}\, dx$$

So we end with the equation,

$$\int_{-1}^{1} u^{n+1} w\, dx = \int_{-1}^{1} u^n w\, dx + \Delta t \int_{-1}^{1} \left[\frac{(u^n)^2}{2} - \varepsilon u_x^n + \frac{\Delta t}{2}\left(2\varepsilon u^n u_{xx}^n - (u^n)^2 u_x^n + \varepsilon (u_x^n)^2\right)\right] w_x\, dx$$

$$+ \frac{(\Delta t)^2 \varepsilon^2}{2} \int_{-1}^{1} u_{xx}^n w_{xx}\, dx$$

TWO - STEPS TAYLOR - GALERKIN METHOD

We introduce an intermediate time:

$$\begin{cases} u^{n+1/2} = u^n + \frac{\Delta t}{2} u_t^n \\ u^{n+1} = u^n + \Delta t \, u_t^{n+1/2} \end{cases} \longrightarrow \begin{cases} u^{n+1/2} = u^n - \frac{\Delta t}{2} f_x(u^n) \\ u^{n+1} = u^n - \Delta t \, f_x(u^{n+1/2}) \end{cases}$$

So the method consists in (at each iteration) firstly, find $u^{n+1/2}$ to, secondly, evaluate $f_x(u^{n+1/2})$ to find u^{n+1}.

WEAK FORM (for both equations) we consider $w, \psi \in H^1([-1,1])$ such that $w(-1) = w(1) = 0$ and $\psi(-1) = \psi(1) = 0$. $(*)_2$

$$\begin{cases} \int_{-1}^{1} u^{n+1/2} w \, dx = \int_{-1}^{1} u^n w \, dx - \frac{\Delta t}{2} \int_{-1}^{1} f_x(u^n) w \, dx \\ \int_{-1}^{1} u^{n+1} \psi \, dx = \int_{-1}^{1} u^n \psi \, dx - \Delta t \int_{-1}^{1} f_x(u^{n+1/2}) \psi \, dx \end{cases}$$

- Integrating by parts:

$$\begin{cases} \int_{-1}^{1} f_x(u^n) w \, dx = [f(u^n(1)) \overset{0}{\underset{(*)_2}{\overline{w(1)}}} - f(u^n(-1)) \overset{0}{\underset{(*)_2}{\overline{w(-1)}}}] - \int_{-1}^{1} f(u^n) w_x \, dx \\ \int_{-1}^{1} f_x(u^{n+1/2})\psi \, dx = [f(u^{n+1/2}(1)) \overset{0}{\underset{(*)_2}{\overline{\psi(1)}}} - f(u^{n+1/2}(1)) \overset{0}{\underset{(*)_2}{\overline{\psi(-1)}}}] - \int_{-1}^{1} f(u^{n+1/2}) \psi_x \, dx \end{cases}$$

thus, the weak form of the equations will be

$$\begin{cases} \int_{-1}^{1} u^{n+1/2} w \, dx = \int_{-1}^{1} u^n w \, dx + \frac{\Delta t}{2} \int_{-1}^{1} f(u^n) w_x \, dx \\ \int_{-1}^{1} u^{n+1} \psi \, dx = \int_{-1}^{1} u^n \psi \, dx + \Delta t \int_{-1}^{1} f(u^{n+1/2}) \psi_x \, dx \end{cases}$$

$$\longrightarrow \begin{cases} \int_{-1}^{1} u^{n+1/2} w \, dx = \int_{-1}^{1} u^n w \, dx + \frac{\Delta t}{2} \int_{-1}^{1} \left[\frac{(u^n)^2}{2} - \varepsilon u_x^n \right] w_x \, dx \\ \int_{-1}^{1} u^{n+1} \psi \, dx = \int_{-1}^{1} u^n \psi \, dx + \Delta t \int_{-1}^{1} \left[\frac{(u^{n+1/2})^2}{2} - \varepsilon u_x^{n+1/2} \right] \psi_x \, dx \end{cases}$$

So in the One-Step method we have as higher derivatives term: $\int_{-1}^{1} u_{xx}^n w_{xx}^n \, dx$

and we need as conditions on w: $w(\underline{x}) = w_x(\underline{x}) = 0$ for $\underline{x} \in \Gamma_D$

in the two-Step method we have as higher derivatives term: $\int u_x^n w_x^n \, dx$ or $\int u_x^{n+1/2} \psi_x^* \, dx$

an we need just the condition $\psi(\underline{x}) = w(\underline{x}) = 0$ for $\underline{x} \in \Gamma_D$