Masters In Computational Mechanics
Finite Elements in Fluid
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Assignment 3

- Write the one-step and two Step Taylor-Galerkin method for the perturbed Burger's equation

$$u_t + uu_x = \epsilon u_{xx} \quad , \text{for } (x,t) \in (-1,1) \times [0,T]$$
$$u^{\epsilon}(x,0) = u_0(x) \quad , \text{for } x \in [-1,1]$$
$$u^{\epsilon}(-1,t) = u(1,t) = 0 \quad , \text{for } t \in [0,T]$$

→ a] One-Step Taylor-Galerkin

as we know for second order expansion in time:

$$u^{n+1} = u^n + \Delta t\, u_t^n + \frac{1}{2}(\Delta t)^2 u_{tt}^n$$

given equation,

$$u_t = -f_x + \epsilon u_{xx} \quad \text{-------- (1)}$$
$$\therefore \quad u_{tt} = -f_{xt} + \epsilon u_{xxt}$$
$$= -f_{tx} + \epsilon u_{txx}$$
$$= -(f_u u_t)_x + \epsilon (u_t)_{xx}$$
$$= -(f_u u_t)_x + \epsilon \underbrace{[-f_x + \epsilon u_{xx}]_{xx}}_{\text{From eq (1)}}$$
$$= -(f_u u_t)_x + \epsilon[-f_{xxx} + \epsilon u_{xxxx}]$$

Thus the equation tends to be descrite in time and continuous in space, therefore:

$$\frac{u^{n+1} - u^n}{\Delta t} = u_t^n + \frac{1}{2}\Delta t\, u_{tt}^n$$

$$\therefore \frac{U^{n+1} - U^{n}}{\Delta t} = -f_x + \epsilon U_{xx} + \frac{1}{2}\Delta t \Big[-(f_u U_t)_x + \epsilon(-f_{xxx} + \epsilon U_{xxxx}) \Big]$$

introducing the function w and integrating over the domain

$$\int_{-1}^{+1} w \frac{U^{n+1} - U^{n}}{\Delta t} = \int_{-1}^{+1} w\,(-f_x + \epsilon U_{xx})\, dx +$$

$$+ \frac{\Delta t}{2}\left[\int_{-1}^{1} w \Big[-(f_u U_x)_x + \epsilon(-f_{xxx} + \epsilon U_{xxxx}) \Big] dx \right]$$

By using integration by-parts:

- $-\int_{-1}^{1} w f_x = w f - \int_{-1}^{1} f_x w_x \, dx$
- $\int_{-1}^{1} w \epsilon U_{xx} \, dx = \epsilon w U_x - \int_{-1}^{1} \epsilon U_x w_x \, dx$
- $\int_{-1}^{1} w [f_u(-f_{xx} + \epsilon U_{xx})] dx = w f_u(-f_x + \epsilon U_x) - \int_{-1}^{1} f_u(-f_x + \epsilon U_{xx}) w_x \, dx$
- $\int_{-1}^{1} w(-f_{xxx} + \epsilon U_{xxxx}) dx = w(-f_{xx} + \epsilon U_{xxx}) - \int_{-1}^{1} (-f_{xx} + \epsilon U_{xxx}) w_x dx$

Substituting all the integrated values we get,

$$\int_{-1}^{1} w \frac{U^{n+1}-U^{n}}{\Delta t}\,dx = wf - \int_{-1}^{1} f_u W_x\,dx + \epsilon w U_x - \int_{-1}^{1} \epsilon U_x W_x\,dx$$

$$+ \frac{\Delta t}{2}\Big[- \Big[w f_u(-f_x + \epsilon U_{xx}) \Big]_{-1}^{1} - \int f_u(-f_x + \epsilon U_{xx})$$

$$+ \epsilon w(-f_{xx} + \epsilon U_{xxx}) - \int_{-1}^{1} \epsilon(-f_{xx} + \epsilon U_{xxx}) W_x\,dx \Big]$$

$\therefore$ Weak form of the solution is

$$\int_{-1}^{1} w \frac{U^{n+1}-U^{n}}{\Delta t}\,dx = - \int_{-1}^{1} w_x (f_u + \epsilon U_x)\,dx$$

$$+ \frac{\Delta t}{2}\Big[- \int_{-1}^{1} w_x \Big[f_u(-f_x + \epsilon U_{xx}) - \epsilon f_{xx} + \epsilon^2 U_{xxx} \Big] \Big]$$

$$+ w\Big[f_u + \epsilon U_x\Big]\Big|_{-1}^{1} + \frac{\Delta t}{2} w\Big[-f_u(-f_x + \epsilon U_{xx})$$

$$- f_{xx} + \epsilon U_{xxx} \Big]\Big|_{-1}^{1}$$

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b] Two - Step ~~Galerki~~ Taylor - Galerkin

- For two-step time form with intermediate step,

$$u^{n+1/2} = u^n + \frac{\Delta t}{2} u_t^n$$

$$u^{n+1} = u^n + \Delta t\, u_t^{n+1/2}$$

$$\therefore \quad \frac{u^{n+1/2} - u^n}{\Delta t} = \frac{u_t^n}{2}$$

substituting given equation

$$\frac{u^{n+1/2} - u^n}{\Delta t} = \frac{1}{2}\left(-f_x + \epsilon u_{xx}\right)^{n+1/2}$$

Introducing the test function w and integrating over the domain

$$\int_{-1}^{1} w \frac{u^{n+1} - u^n}{\Delta t} = \int_{-1}^{1} w\left(-f_x^{n+\frac{1}{2}} + \epsilon u_{xx}^{n+\frac{1}{2}}\right) dx$$

$$= w\left(-F_u^{n+\frac{1}{2}} + \epsilon u_x^{n+\frac{1}{2}}\right) - \int_{-1}^{1} w_x\left(-f_u^{n+\frac{1}{2}} + \epsilon u_x^{n+\frac{1}{2}}\right) dx$$

- Therefore weak form of two-step Taylor Galerkin

$$\int_{-1}^{1} w \frac{u^{n+1} - u^n}{\Delta t} = -\int_{-1}^{1} w_x\left(-f_u^{n+\frac{1}{2}} + \epsilon u_x^{n+1/2}\right) + \left[w\left(-f_u^{n+1/2} + \epsilon u_x^{n+1/2}\right)\right]_{-1}^{1}$$

Thus, we can conclude time descretization in one-step method is complex then two step method, but we obtain final value in one-step where as in two step we get value at intermediate step.