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$$\begin{cases} u_t^\epsilon + u^\epsilon u_x^\epsilon = \epsilon u_{xx}^\epsilon & \text{for } (x,t) \in [-1,1] \times [0,T] \\ u^\epsilon(x,0) = u_0(x) & \text{for } x \in [-1,1] \\ u^\epsilon(-1,t) = u^\epsilon(1,t) = 0 & \text{for } t \in [0,T] \end{cases}$$

$$\begin{cases} u_t^\epsilon = -u^\epsilon u_x^\epsilon + \epsilon u_{xx}^\epsilon \\ u_{tt}^\epsilon = -(u_t^\epsilon u_x^\epsilon + u^\epsilon u_{x_t}^\epsilon) + (\epsilon u_{xx}^\epsilon)_t \end{cases}$$

$$\hookrightarrow u_t^\epsilon = -u^\epsilon D u^\epsilon + \epsilon D^2 u^\epsilon \rightarrow -u^\epsilon D u^\epsilon = u_t^\epsilon - \epsilon D^2 u^\epsilon$$

$$u_{tt}^\epsilon = -(u_t^\epsilon D u^\epsilon + u^\epsilon D u_t^\epsilon) + \epsilon D^2 u_t^\epsilon$$

$$u_{tt}^\epsilon = -\left[(-u^\epsilon D u^\epsilon + \epsilon D^2 u^\epsilon) D u^\epsilon + u^\epsilon D \cdot (-u^\epsilon D u^\epsilon + \epsilon D^2 u^\epsilon)\right] + \epsilon D^2 u_t^\epsilon$$

$$u_{tt}^\epsilon = -\left[-(u^\epsilon \cdot D)^2 u^\epsilon + (\epsilon D^2) u^\epsilon D u^\epsilon \cancel{+} \epsilon D^2 u_t^\epsilon + (-u^\epsilon D u^\epsilon + \epsilon D^2 u^\epsilon) D u^\epsilon\right]$$

$$u_{tt}^\epsilon = (u^\epsilon \cdot D)^2 u^\epsilon - (\epsilon D^2) u^\epsilon D u^\epsilon + \epsilon D^2 u_t^\epsilon + (u^\epsilon D u^\epsilon - \epsilon D^2 u^\epsilon) D u^\epsilon$$

$$+ \epsilon D^2 (u_t^\epsilon - \epsilon D^2 u^\epsilon]$$

$$u_{tt}^\epsilon = (u^\epsilon \cdot D)^2 u^\epsilon + \epsilon D^2 u_t^\epsilon - \epsilon^2 D^4 u^\epsilon + \epsilon D^2 u_t^\epsilon + u^\epsilon (D u^\epsilon)^2 - \cancel{\epsilon D (D u)^2} - \epsilon \cdot D^3 u^{\epsilon 2}$$

$$u_{tt}^\epsilon = 2(u^\epsilon \cdot D)^2 u^\epsilon + 2\epsilon D^2 u_t^\epsilon - \epsilon^2 D^4 u^\epsilon - \epsilon D^3 u^{\epsilon 2}$$

replace $u_t^\epsilon = \dfrac{u^{t+1} - u^t}{\Delta T}$

$$\frac{u^{t+1} - u^t}{\Delta t} = u_t^\epsilon + \frac{1}{2} \Delta t\, u_{tt}^\epsilon$$

$$\frac{u^{t+1} - u^t}{\Delta t} = -u^\epsilon D u^\epsilon + \epsilon D^2 u^\epsilon + \frac{1}{2} \Delta T \left[2(u^\epsilon \cdot D)^2 u^\epsilon + 2\epsilon D^2 \left(\frac{u^{t+1} - u^t}{\Delta T}\right) - \epsilon^2 D^4 u^\epsilon - \epsilon D^3 u^{\epsilon 2}\right]$$

$$\boxed{(1 - \Delta T \epsilon D^2)\left(\frac{u^{t+1} - u^t}{\Delta T}\right) = -u^t D u^t + \epsilon D^2 u^t + \frac{1}{2} \Delta T \left[2(u^\epsilon \cdot D)^2 u^t - \epsilon^2 D^4 u^t - \epsilon D^3 u^{t 2}\right]}$$

$\uparrow$ 1-step TG discrete in time

2-step TG

$$u^{t+1} = u^t + \Delta t\left(u^t + \frac{\Delta t}{2} u_t^t\right)_t$$

$$u^{t+\frac{1}{2}} = u^t + \frac{\Delta t}{2} u_t^t \qquad \Bigg| \qquad u_t^t = -u^t \nabla u^t + \epsilon \nabla^2 u^t$$

$$u^{t+1} = u^t + \Delta t\, u_t^{t+\frac{1}{2}} \qquad \Bigg| \qquad u_t^{t+\frac{1}{2}} = -u^{t+\frac{1}{2}} \nabla u^{t+\frac{1}{2}} + \epsilon \nabla^2 u^{t+\frac{1}{2}}$$

$$\left[\begin{array}{l} u^{t+\frac{1}{2}} = u^t + \frac{\Delta t}{2}\left(-u^t \nabla u^t + \epsilon \nabla^2 u^t\right) \\[2ex] u^{t+1} = u^t + \Delta t\left(-u^{t+\frac{1}{2}} \nabla u^{t+\frac{1}{2}} + \epsilon \nabla^2 u^{t+\frac{1}{2}}\right) \end{array}\right.$$