Màster en Mètodes Numèrics **Aitor Bazán Escoda**

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Finite Elements in Fluids

Homework 3: Unsteady convection problems

Problem statement

Case 1. Pure convection

$$\begin{cases} u_t + a u_x = 0 \; x \in (0,1), t \in (0,0.6] \\ u(x,0) = u_0(x) \; x \in (0,1) \\ u(0,t) = 0 \; t \in (0,0.6] \end{cases}$$

$$u_0(x) \begin{cases} \frac{1}{2}\left(1 + \cos\left(\frac{\pi(x - x_0)}{\sigma}\right)\right) & if \; |x - x_0| \leq \sigma \\ 0 & otherwise \end{cases}$$

Since $a = 1, x_0 = 0.2, \sigma = 0.12, \Delta x = 2 \cdot 10^{-2}$

Next it is shown the solution of the pure convection case for different explicit and implicit time-stepping methods. To get better insight on the stability of these methods, they have been implemented with the lumped mass matrix formulation besides of the consistent mass matrix which was already set.

The general purpose of implementing the lumped mass matrix is meant to be done to update the solution in a fully explicit way or make an implicit algorithm more robust and efficient.

In the case presented, they are used the row-sum mass lumping technique and the diagonalization of the mass matrix which coincide for the use of linear elements.

The number of elements is set to 50 since

$$\frac{1}{2 \cdot 10^{-2}}$$

To check reliability on the worked cases Courant = 0.6 to check if the figures obtained are the same as the ones given in the slides. This is not part of the exercises.

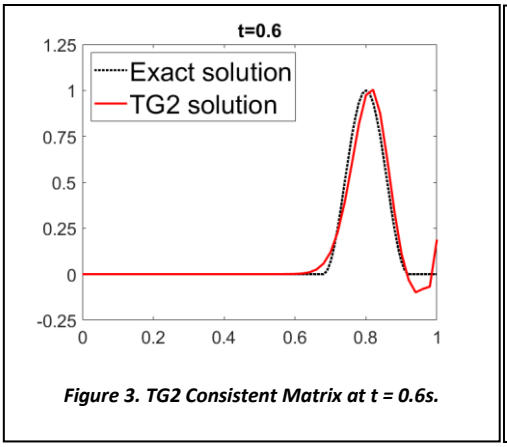

Figure 3. TG2 Consistent Matrix at t = 0.6s.

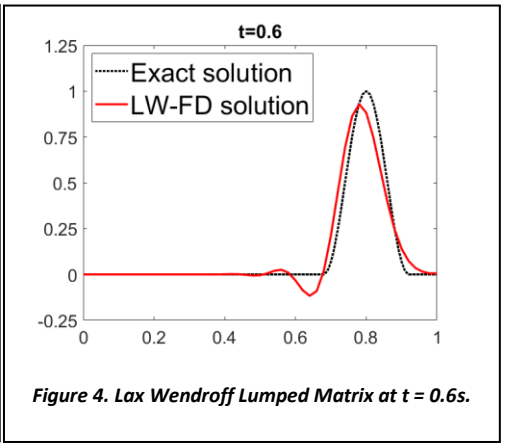

Figure 4. Lax Wendroff Lumped Matrix at t = 0.6s.

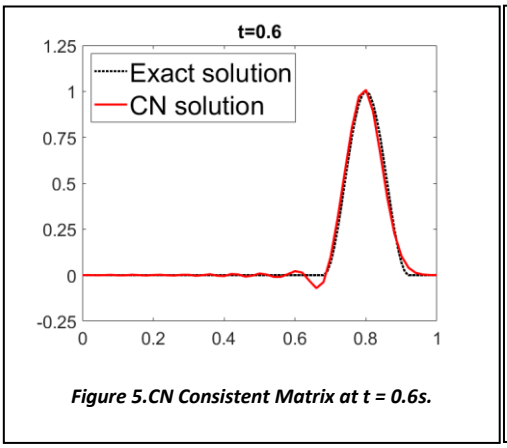

Figure 5.CN Consistent Matrix at t = 0.6s.

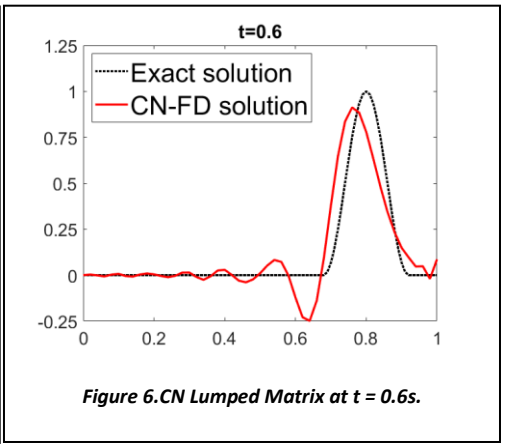

Figure 6.CN Lumped Matrix at t = 0.6s.

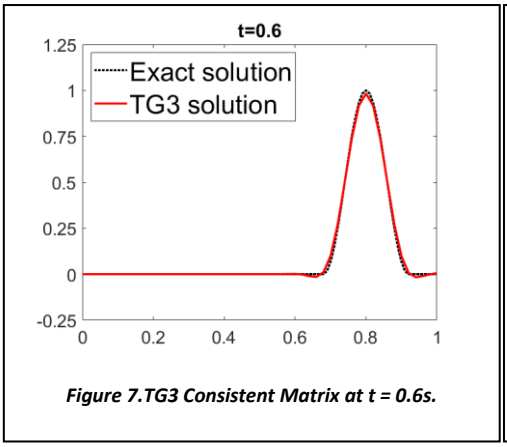

Figure 7.TG3 Consistent Matrix at t = 0.6s.

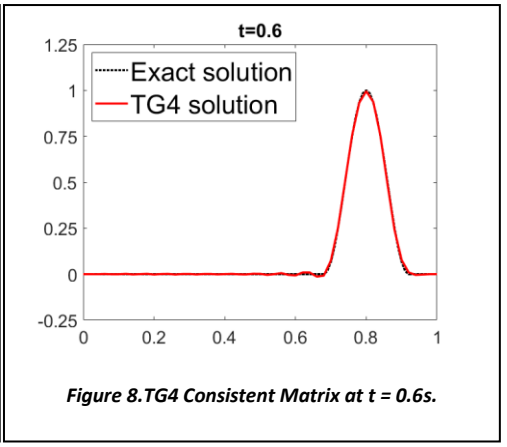

Figure 8.TG4 Consistent Matrix at t = 0.6s.

As it can be seen in the figures above, the solution by means of Lax-Wendroff (TG2) with consistent matrix exhibits better phase accuracy than the scheme for Lax-Wendroff with diagonal matrix. The stability limit for second order Galerkin is $C^2 \leq 1/3$. When Lax-Wendroff is performed with diagonal matrix is stable up to $C^2 = 1$ and possesses the so-called CFF property. That means that exact nodal solution is obtained on a uniform mesh when $C^2 = 1$. Therefore, as it is shown in the figure:

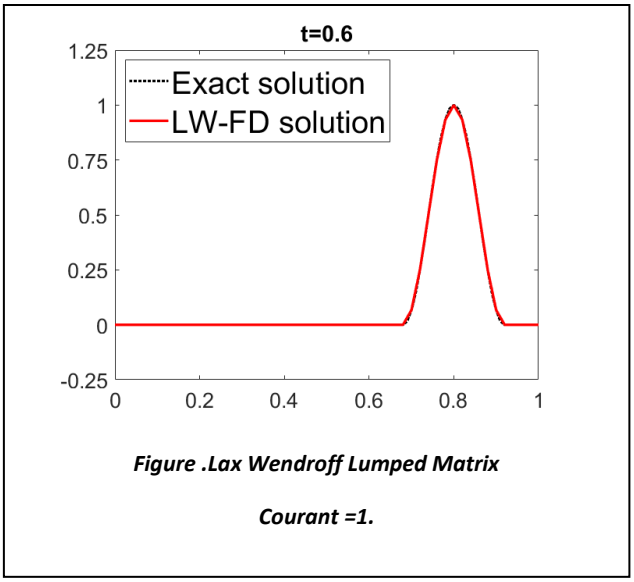

Figure .Lax Wendroff Lumped Matrix

Courant =1.

On the other hand, Lax Wendroff with diagonal mass matrix has predominant lagging phase error, except for large wave numbers, this is when C > ½. Therefore, to illustrate the behaviour of the explained we have:

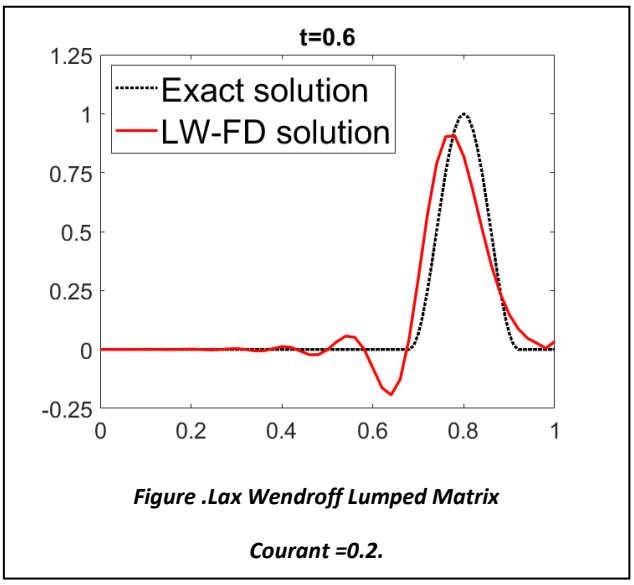

Figure .Lax Wendroff Lumped Matrix

Courant =0.2.

Taylor Galerkin 3 it exhibits a uniform phase accuracy over the interval 0 < C < 1 while TG4 has excellent performance at all numbers of Courant.

So, as a summary, it needs to be noted that methods with consistent matrix have better phase accuracy than the ones with diagonal mass when keeping C in the stability range.

The lumped matrix consists on adding to the diagonal terms, the sum of the rows.

```
function [Masslumped]=Lumper(M)
sizeM=size(M);
Masslumped=zeros(sizeM(1),sizeM(2));
for i=1:sizeM(1)
    Lumped=0;
    for j=1:sizeM(2)
        Lumped=Lumped+M(i,j);
    end
    Masslumped(i,i)=Lumped;
end
```

It is also mandatory to implement the choice for lumped matrix use in the main function.

```
    if lumped_m == 1
        M(Te,Te) = M(Te,Te) + Me;
        Masslumped=Lumper(M);
    else
        M(Te,Te) = M(Te,Te) + Me;
    end
```

Some research has driven to find an expression that works fine when using linear elements. This is the case of the diagonalization of the mass matrix.

```
        if lumped_m == 1
        Me = Me + w_ig*diag((N_ig'*N_ig)*unos);
        else
        Me = Me + w_ig*(N_ig'*N_ig);
        end
        Ke = Ke + w_ig*(Nx_ig'*Nx_ig);
        Ce = Ce + w_ig*(N_ig'*Nx_ig);
    end
```

To show how the expressions for the already discretized in time and in space are typed, we refer to the following:

```
function [A,B,methodName] = System(method,M,K,C,a,dt,Mlumped)


switch method
    case 1 % Lax-Wendroff + Galerkin
        A = M;
        B = (a*dt*C' - 0.5*(dt^2)*(a^2)*K);
        methodName = 'TG2';
    case 2 % Lax-Wendroff with lumped mass matrix + Galerkin
        A = M;
        B = (a*dt*C' - 0.5*(dt^2)*(a^2*K));
        methodName = 'LW-FD';
    case 3 % Crank-Nicolson + Galerkin
        A = M - 1/2*a*dt*C';
        B = a*dt*C';
        methodName = 'CN';
    case 4 % Crank-Nicolson with lumped mass matrix + Galerkin
        A = M - 1/2*a*dt*C';
        B = (a*dt*C');
        methodName = 'CN-FD';
    case 5 % Taylor-Galerkin third-order
        A = M + (1/6)*(dt^2)*(a^2)*K;
        B = (dt*a*C' - (1/2)*(dt^2)*(a^2)*K);
        methodName = 'TG3';
    case 6 % Taylor-Galerkin 4th-order
        A = M + (1/2)*dt*a*C - (1/12)*(dt^2)*(a^2)*K;
        B = -dt*a*C;
        methodName = 'TG4';
    otherwise
        error('not available method')
end
```

Problem statement

Case 2. Propagation of a steep front

$$\begin{cases} u_t + au_x = 0 \; x \in (0,1), t \in (0,0.6] \\ u(x,0) = u_0(x) \; x \in (0,1) \\ u(0,t) = 1 \; t \in (0,0.6] \end{cases}$$

$$u_0(x) \begin{cases} 1 & if x \leq 0.2 \\ 0 & otherwise \end{cases}$$

With the conditions: $a = 1, \Delta x = 2 \cdot 10^{-2}, \Delta t = 1.5 \cdot 10^{-2}$

Throughout this date, we can conclude that the Courant number is 0.75.

And, also, the number of time steps is 40 due to:

$$\frac{0.6}{1.5 \cdot 10^{-2}}$$

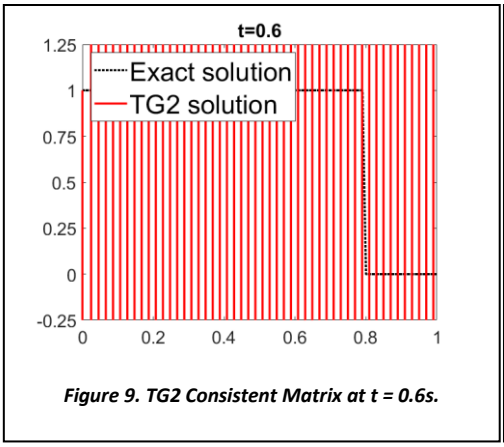

Figure 9. TG2 Consistent Matrix at t = 0.6s.

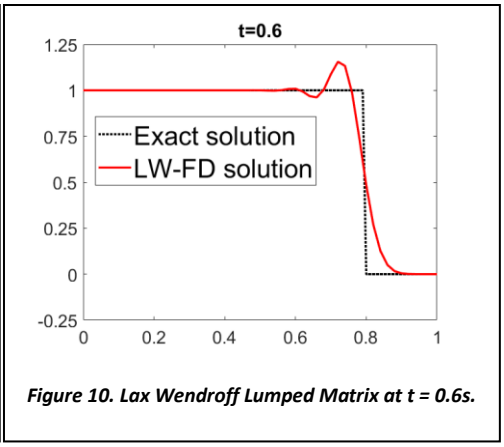

Figure 10. Lax Wendroff Lumped Matrix at t = 0.6s.

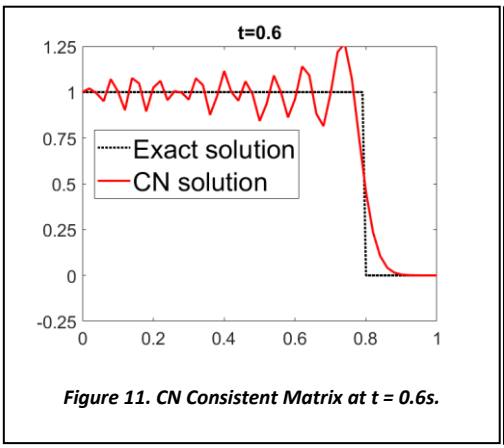

Figure 11. CN Consistent Matrix at t = 0.6s.

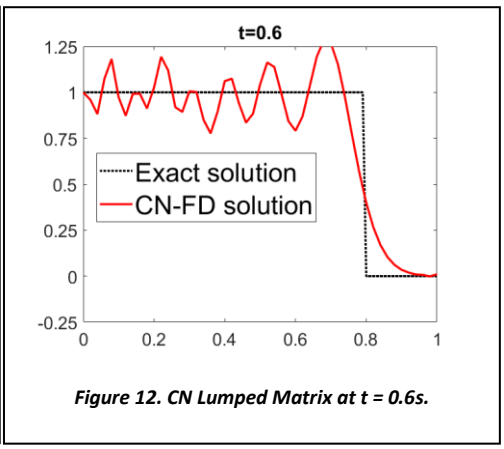

Figure 12. CN Lumped Matrix at t = 0.6s.

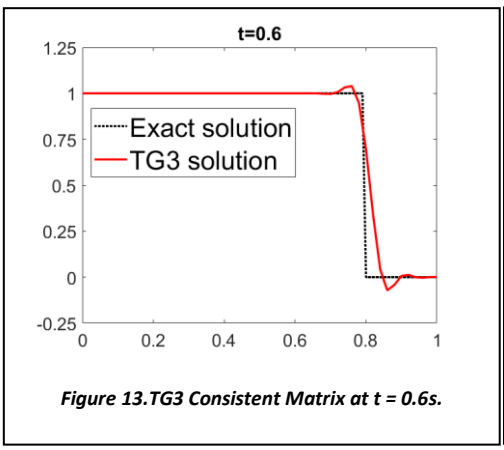

Figure 13.TG3 Consistent Matrix at t = 0.6s.

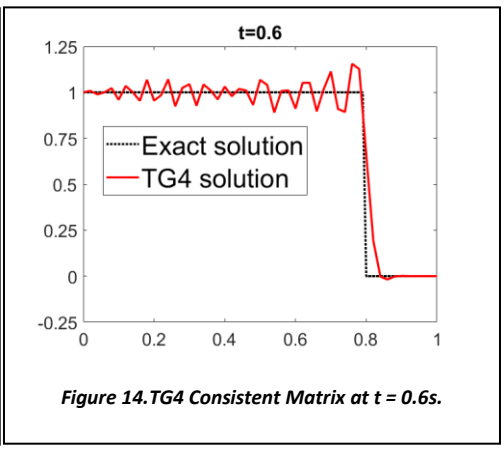

Figure 14.TG4 Consistent Matrix at t = 0.6s.

1. Compute Courant number: $\frac{0.6\cdot\frac{1}{\Delta t}}{\frac{1}{\Delta x}} = \frac{0.6\cdot\frac{1}{1.5\cdot 10^{-2}}}{\frac{1}{2\cdot 10^{-2}}} = 0.75$
2. The solution for CN by means of both consistent and lumped matrices is not accurate. It introduces spurious oscillations due to Galerkin formulation. Also, these residual oscillations remain at the front since Crank Nicolson is not such a monotone scheme. By using nonlinear viscosity added at the front to improve the scheme locally first order accurate.
3. The solution with Lax-Wendroff (TG2) introduces even more instabilities due to be, the computed Courant number, overcoming the expected limit number for such scheme which is $C^2 \leq 1/3$. On the other hand, the solution could be improved by working with diagonal mass matrix (lumped mass matrix). This way, the stability range for Courant

number gets broaden. However, there are still spurious oscillations. Stability for this method is commented in first exercise.

4. In the case of third-order Galerkin, it comes out a solution in which the oscillations are mainly removed. It gets better accuracy than Lax Wendroff with diagonal matrix however, due to Galerkin formulation, steep functions cannot be shaped. The extra terms compared to TG2 blur the possible spurious oscillations of that method.