**Finite Elements in Fluids** **February 20, 2019**
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**Assignment II:**

**Steady convection-diffusion equation**

The strong form of the problem is defined as it follows:

$$
\begin{aligned}
&a u_x - \nu u_{xx} = s(x) \qquad && \text{in [0,L]} \\
&u = 0 && \text{at x=0 and x=L}
\end{aligned} \tag{1}
$$

where:

$a$ = Convection velocity
$u$ = the scalar unknown
$\nu$ = is the coefficient of diffusivity
$s$ = is the source term

The weak form associated with this model problem is, after integration by parts of the diffusion term, given by:

$$\int_0^L (w\, a\, u_x + w_x + \nu u_x)\, dx = \int_0^L w\, s\, dx \tag{2}$$

The weak form will now be discretized using a uniform mesh of linear elements of size h.

## 1 Galerkins method

### 1.1 Linear approximation in 1D

A linear element in 1D is defined by two nodes, n = 2, locally denoted as 1 and 2. The shape functions of a linear element are given by:

$$\begin{cases} N_1(\xi) = 1/2(1-\xi) \\ N_2(\xi) = 1/2(1+\xi) \end{cases} \tag{3}$$

where $\xi$ is the normalized coordinate, $1 \leq \xi \leq +1$. As usual, at any interior point of the element one has $u(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2$, and $x(\xi) = N_1(\xi)x_1 + N_2(\xi)x_2$.

If we express the problem in matrix form we have that:

$$(C + K)u = f \tag{4}$$

where:

$C$ = Convection matrix
$K$ = diffusion matrix
$u$ = the vector of the unknown nodal values
$f$ = the vector that considers the contribution of the source term s

We can define then C and K in terms of the nodal numbers 1 and 2:

$$C^e = a \int_{\Omega_e} \begin{pmatrix} N_1 \frac{dN_1}{dx} & N_1 \frac{dN_2}{dx} \\ N_2 \frac{dN_1}{dx} & N_2 \frac{dN_2}{dx} \end{pmatrix} dx = \frac{a}{2} \begin{pmatrix} -1 & +1 \\ -1 & +1 \end{pmatrix} \tag{5}$$

$$K^e = \nu \int_{\Omega_e} \begin{pmatrix} \frac{dN_1}{dx}\frac{dN_1}{dx} & \frac{dN_1}{dx}\frac{dN_2}{dx} \\ \frac{dN_2}{dx}\frac{dN_1}{dx} & \frac{dN_2}{dx}\frac{dN_2}{dx} \end{pmatrix} dx = \frac{\nu}{2} \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix} \tag{6}$$

Furthermore, interpolating the source term s by means of the element shape functions, namely $s(\xi) = N_1(\xi)s_1 + N_2(\xi)s_2$ , one finds that the components of the load vector f in (4) are obtained from element contributions of the form:

$$f^e = \int_{\Omega_e} [N_1(N_1 s_1 + N_2 s_2), N_2(N_1 s_1 + N_2 s_2)]^T dx \tag{7}$$

With these results, and assembling in the usual finite element manner the contributions emanating from both elements to which a given node belongs, one finds that the Galerkin method delivers the following discrete equation at an interior node j:

$$a(\frac{u_{j+1} - u_{j-1}}{2h}) - \nu(\frac{u_{j+1} - 2u_j - u_{j-1}}{h^2}) = \frac{1}{6}(s_{j-1} + 4s_j + s_{j+1}) \tag{8}$$

It is useful to introduce the mesh Pèclet number:

$$P_e = \frac{ah}{2\nu} \tag{9}$$

which expresses the ratio of convective to diffusive transport. This allows us to rewrite the discrete equation (8) in the form

$$\frac{a}{2h}(\frac{P_e - 1}{P_e}u_{j+1} + \frac{2}{P_e}u_j - \frac{P_e + 1}{P_e}u_{j-1}) = \frac{1}{6}(s_{j-1} + 4s_j + s_{j+1}) \tag{10}$$

### 1.2 Quadratic elements in 1D

We consider a generic element with end nodes 1 and 3, and a mid-side node 2. With reference to the normalized coordinate $-1 \le \xi \le +1$ the shape functions of the element are

$$\begin{cases} N_1(\xi) = \tfrac{1}{2}\xi(\xi - 1) \\ N_2(\xi) = 1 - \xi^2 \\ N_2(\xi) = \tfrac{1}{2}\xi(\xi + 1) \end{cases} \tag{11}$$

It follows that at any interior point of an element, one has:

$$u(\xi) = N_1(\xi)u_1 + N_2(\xi)u_2 + N_3(\xi)u_3 \tag{12}$$

$$x(\xi) = N_1(\xi)x_1 + N_2(\xi)x_2 + N_3(\xi)x_3 \tag{13}$$

With these interpolation functions the convection, $C_e$ , and diffusion, $K_e$, matrices of the quadratic element are:

$$C^e = a \int_{\Omega_e} \begin{pmatrix} N_1 \dfrac{dN_1}{dx} & N_1 \dfrac{dN_2}{dx} & N_1 \dfrac{dN_3}{dx} \\ N_2 \dfrac{dN_1}{dx} & N_2 \dfrac{dN_2}{dx} & N_2 \dfrac{dN_3}{dx} \\ N_3 \dfrac{dN_1}{dx} & N_3 \dfrac{dN_2}{dx} & N_3 \dfrac{dN_3}{dx} \end{pmatrix} dx = \frac{a}{2} \begin{pmatrix} -1 & 4/3 & -1/3 \\ -4/3 & 0 & 4/3 \\ 1/3 & -4/3 & 1 \end{pmatrix} \tag{14}$$

$$K^e = \nu \int_{\Omega_e} \begin{pmatrix} \dfrac{dN_1}{dx}\dfrac{dN_1}{dx} & \dfrac{dN_1}{dx}\dfrac{dN_2}{dx} & \dfrac{dN_1}{dx}\dfrac{dN_3}{dx} \\ \dfrac{dN_2}{dx}\dfrac{dN_1}{dx} & \dfrac{dN_2}{dx}\dfrac{dN_2}{dx} & \dfrac{dN_2}{dx}\dfrac{dN_3}{dx} \\ \dfrac{dN_3}{dx}\dfrac{dN_1}{dx} & \dfrac{dN_3}{dx}\dfrac{dN_2}{dx} & \dfrac{dN_3}{dx}\dfrac{dN_3}{dx} \end{pmatrix} dx = \frac{\nu}{6h} \begin{pmatrix} 7 & -8 & 1 \\ -8 & 1 & -8 \\ 1 & -8 & 7 \end{pmatrix} \tag{15}$$

As with the linear element, we interpolate the source term s(x) in (2.16) by means of the element shape functions, $s(\xi) = N_1(\xi)s_1 + N_2(\xi) + N_3(\xi)s_3$ , one finds that the components of the load vector f in (4) are obtained from element contributions of the form:

$$f^e = \int_{\Omega_e} [N_1(N_1s_1 + N_2s_2 + N_3s_3), N_2(N_1s_1 + N_2s_2 + N_3s_3), N_3(N_1s_1 + N_2s_2 + N_3s_3)]^T dx \tag{16}$$

With the element matrices presented in (14) and (15), and assembling in the usual finite element manner,the Galerkin method delivers two types of nodal equations representing the discrete counterpart of the convection-diffusion equation:

1) At a mid-side node i the discrete equation is obtained in the form:

$$a(\frac{u_{i+1}-u_{i-1}}{2h}) - \nu(\frac{u_{i+1}-2u_i-u_{i-1}}{h^2}) = \frac{1}{10}(s_{i-1}+8s_i+s_{i+1}) \tag{17}$$

2) At the corner (inter-element) node j the discrete equation involves a stencil of five nodes and reads:

$$a[2(\frac{u_{j+1}-u_{j-1}}{2h})-(\frac{u_{j+2}-u_{j-2}}{4h})]-\nu[2(\frac{u_{j-1}-2u_j+u_{j+1}}{h^2})-(\frac{u_{j-2}-2u_j+u_{j+2}}{4h^2})] = \frac{1}{10}(-s_{j-2}+2s_{j-1}+8s_j+ \tag{18}$$

## 2 Streamline-upwind method (SU)

Improves the behavior of the Galerkin method in convection-dominated problems adding artificial diffusion to counteract the negative dissipation introduced by the Galerkin formulation.
For a simple case of the linear convection-diffusion problem with a constant source term:

$$au_x - (\nu+\overline{\nu})u_{xx} = 0 \qquad \text{with} \qquad \overline{\nu} = \beta\frac{ah}{2} \tag{19}$$

containing a free parameter $\beta$ $(0 \leq \beta \leq 1)$ which governs the amplitude of the added numerical diffusion.

The corresponding weak form for s=0:

$$\int_0^L (w\,a\,u_x + w_x(\nu+\overline{\nu})u_x)\,dx = 0 \tag{20}$$

For the problem:

$$\begin{aligned} au_x - \nu u_{xx} &= s(x) && \text{in [0,L]} \\ u &= 0 && \text{at x=0 and x=L} \end{aligned}$$

The optimal value of $\beta$ is equal to $cothP_e - \frac{1}{P_e}$ , and the value $\beta = 1$ corresponds to full upwind differencing.

## 3 Streamline Upwind Petrov-Galerkin (SUPG)

The goal of this method is to stabilize the convective term in a consistent manner, that is, ensuring that the exact solution is also a solution of the weak formulation.

Taking:

$$\mathcal{P}(\omega) = a \cdot \nabla\omega \tag{21}$$

Stabilized consistent formulation:

$$a(\omega, u) + c(\omega, u, a) + (\omega, \sigma u) + \boxed{\sum_e \int_{\Omega_e} \mathcal{P}(\omega)\tau\mathcal{R}(u)d\Omega} = (\omega, s) + (\omega, u_N)_{\Gamma N} \tag{22}$$

In case of 1D we have the fallowing equation:

$$\int_0^L (w\,a\,u_x + w_x\nu u_x)\,dx + \boxed{\sum_e \int_{\Omega_e} \tau(aw_x)(au_x - \nu u_{xx} - s)dx} = \int_0^L w\,s\,dx \tag{23}$$

Added diffusion $\rightarrow \overline{\nu} = \beta\dfrac{ah}{2}$

When we are using lineal elements the function becomes:

$$\int_0^L (w\,a\,u_x + w_x\nu u_x)\,dx + \boxed{\sum_e \int_{\Omega_e} \tau(aw_x)(au_x - s)dx} = \int_0^L w\,s\,dx \tag{24}$$

## 4 Galerkin least-squares (GLS)

The GLS technique is defined by imposing that the stabilization term in (14) is an element-by-element weighted least-squares formulation of the original differential equation. This corresponds to the following choice for the operator applied to the test function:

$$\mathcal{P}(\omega) = \mathcal{L}(\omega) = a \cdot \nabla\omega - \nabla \cdot (\nu\nabla\omega) + \sigma\omega \tag{25}$$

SUPG and GLS are identical for convection-diffusion equations with linear elements.

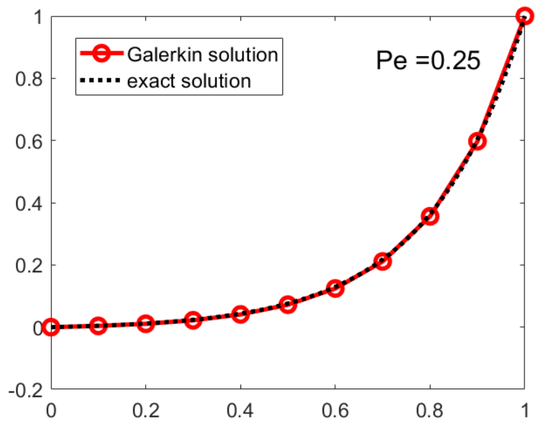
## 5 Analysis

### 5.1 The following equation will be analyzed for different cases:

| | | |
|---|---|---|
| $au_x - \nu u_{xx} = f'$ | | in [0,1] |
| $u(0) = 0$ | $u(1) = 1$ | Boundary conditions |
| $f = 0$ | | Source term |

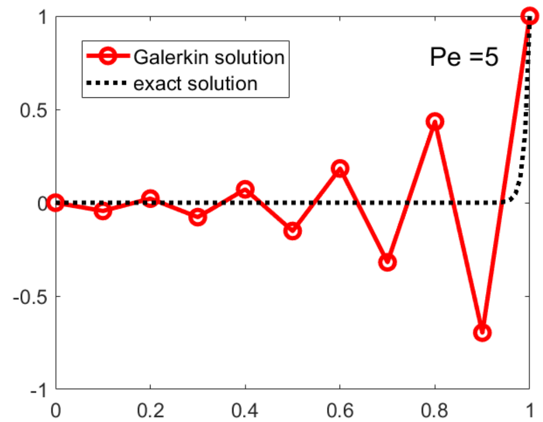
### 5.1.1 Case 1

$$a = 1$$
$$\nu = 0.2$$
$$\text{number of elements} = 10$$



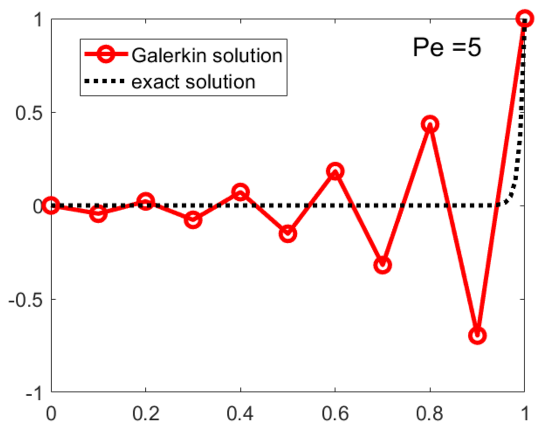

### 5.1.2 Case 2

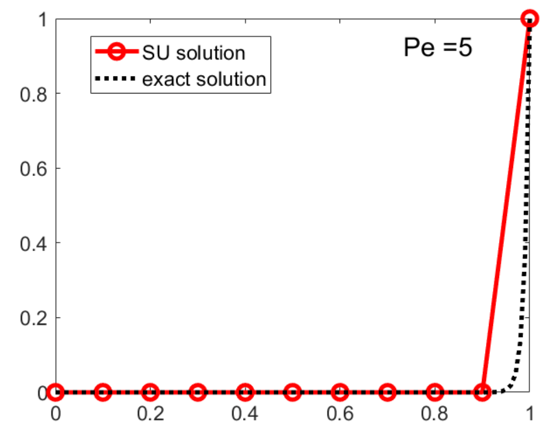
$$a = 20$$
$$\nu = 0.2$$
$$\text{number of elements} = 10$$

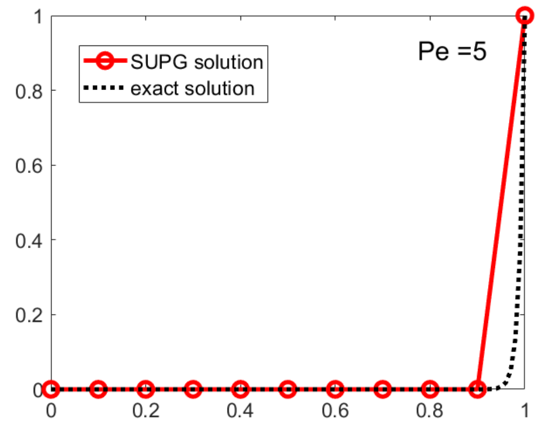



### 5.1.3 Case 3

$$a = 1$$
$$\nu = 0.01$$
$$\text{number of elements} = 10$$

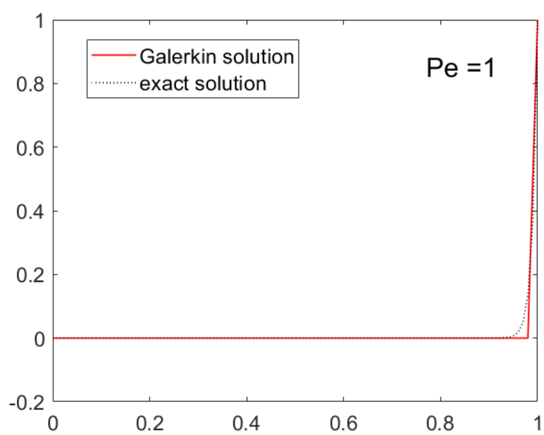





#### 5.1.4 Case 4

$$a = 1$$
$$\nu = 0.01$$
$$\text{number of elements} = 50$$



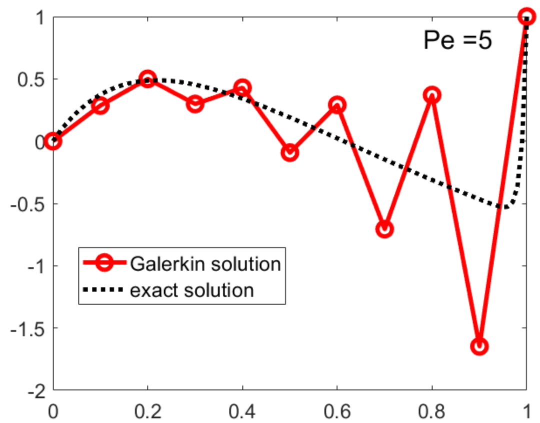

### 5.2 The following equation will be analyzed

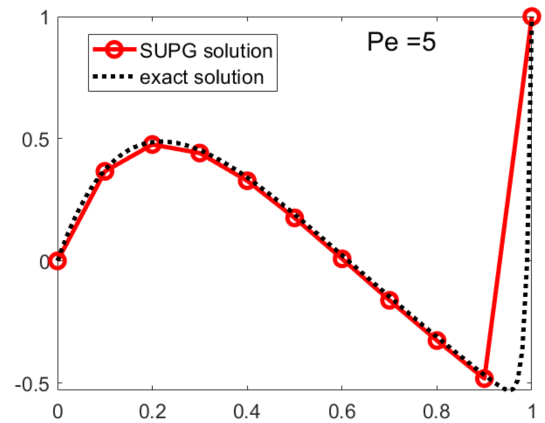
$$\begin{aligned} &au_x - \nu u_{xx} = f' && \text{in } [0,1] \\ &u(0) = 0 \qquad u(1) = 1 && \text{Boundary conditions} \\ &f = 10e^{-5x} - 4e^{-x} && \text{Source term} \end{aligned}$$

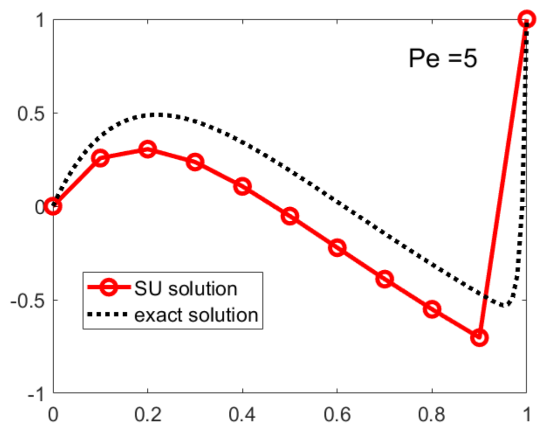
Using the following values:

$$a = 1$$
$$\nu = 0.01$$
$$\text{number of elements} = 10$$

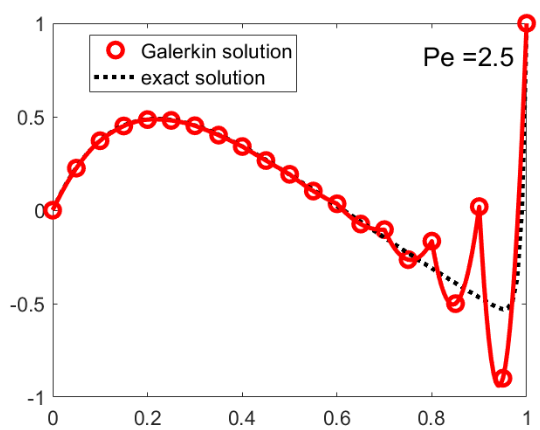
### 5.2.1 Using lineal elements we have:

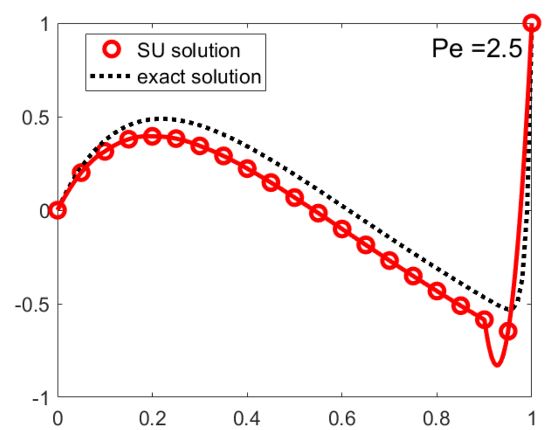


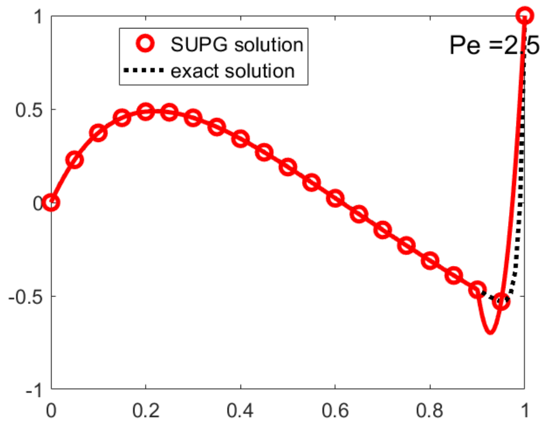



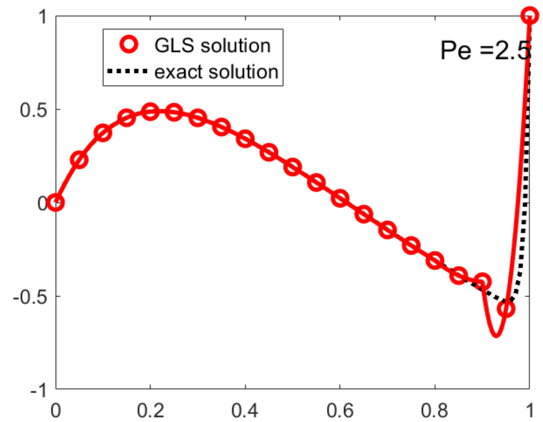




## 5.3 Using quadratic elements we have:








## 6 Conclusion

It is noted that Galerkin's solution is corrupted by non-physical oscillations when Peclet's number is greater than one. The Galerkin method loses its best approximation property when the non-symmetric convection operator dominates the diffusion operator in the transport equation and, consequently, oscillations occur from node to node.

SU will give us the exact solution at nodes when the source term is 0 or a constant value, but when we add a not constant source term the solution the numerical solution represents the trend correctly but has a huge error.

In the case of USPG and GLS the solution in exact for the nodes in both cases, when we have a source that does not have a constant value and when we do not.