<u>Class Homework 2:</u> 1D Unsteady convection transportation

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The equation to be solved correspond to the 1D unsteady convection transportation. The source term is s=0 and there is no Neumann bc.

$$
\begin{aligned}
&u_t + (\boldsymbol{a}\cdot\boldsymbol{\nabla})u = 0 && \text{in } \Omega\times]0,T[,\\
&u(\boldsymbol{x},0) = u_0(\boldsymbol{x}) && \text{on } \Omega \text{ at } t=0,\\
&u = u_D && \text{on } \Gamma_D^{in}\times]0,T[
\end{aligned}
$$

<u>Leap-Frog (LF) implementation:</u>

The next figure shows the line of the function code called "System" where the LF method was implemented.
After discretizing, the expression of the LF method is:

$$\left(w, \frac{u^{n+1}}{2\Delta t}\right) = \left(w, \frac{u^{n-1}}{2\Delta t}\right) + (w, -\boldsymbol{a}\cdot\boldsymbol{\nabla}u^n)$$

Since there is no Neumann bc it was not necessary to integrate by part.

```
case 5 % Leap-Frog method + Galerkin
    A = M;
    B = C;
    methodName = 'LF';
```

$$(w, \frac{u^{n+1}}{2\Delta t}) = \frac{M}{2\Delta t}U^{n+1}$$

M: consistent mass matrix. This expression corresponds to the l.h.s of the equation.

$$M = \int_{\Omega^e} N'Nd\Omega$$

$$(w, -a\cdot\nabla u^n) = Cu^n$$

where

$$C = \int_{\Omega^e} N'(a\nabla N)d\Omega$$

As this method is not self-started, it was necessary to use a start method in order to obtain the first two-time step U^n and U^{n-1}. The Crank-Nicholson method was used as start method.
The figure below shows the implementation of the method explained before.

```
if method == 5 % first two time steps are calulated with CN
for n = 1:nStep
    if method == 5 && n==1
        method = 3;
        [P,Q] = System(method,M,K,C,a,dt);
        P = P(ind_unk,ind_unk);
        Q = Q(ind_unk,ind_unk);
        Du = P\(Q*u(ind_unk,n) + f);
        u(ind_unk,n+1) = u(ind_unk,n) + Du;
        method = 5;
    elseif method == 5 && n==2
        Du = P\(Q*u(ind_unk,n) + f);
        u(ind_unk,n+1) = u(ind_unk,n) + Du;
    else
    u_1 = A\(A*u(ind_unk,n-1)-a*2*dt*B*u(ind_unk,n)+f);
    u(ind_unk,n+1) = u_1;
    end
```

Crank-Nicholson for the first two-time steps

The Leap-Frog method

Two-step explicit Taylor-Galerkin methods (TG3-2S):

The next figure shows the line of the function code called “System” where TG3-2S method was implemented.
It can be seen the implementation of the first step.

```
case 6 % Two-step Third order Taylor-Galerkin + Galerkin
 A = M;
 B = M-(1/3)*dt*a*C-(1/9)*a^2*(dt^2)*K;
 methodName = 'TG3-2S';
```

$(w,\tilde{u}^n) = M\tilde{U}^n$

M: consistent mass matrix. This expression corresponds to the l.h.s of the 1st step.

$$Mu^n - \frac{1}{3}\Delta t a C u^n - \frac{1}{9}\Delta t^2 a^2 K U^n$$

Where

$M = \int_{\Omega^e} N' N d\Omega$ (consistency mass matrix)

$C = \int_{\Omega^e} N'(a\nabla N) d\Omega$ (convection matrix)

$K = \int_{\Omega^e} \nabla N' \nabla N d\Omega$ (stiffness matrix)

Implementation of the 2nd step

```
elseif method == 6
    C = C(ind_unk,ind_unk);
    K = K(ind_unk,ind_unk);
    for n = 1:nStep
        Up = A\(B)*u(ind_unk,n);
        Du = A\(-dt*a*C*u(ind_unk,n)-0.5*dt^2*a^2*K*Up + f);
        u(ind_unk,n+1) = u(ind_unk,n) + Du;
    end
```

Resolution of the 1st Step already implemented

Resolution of the 2nd Step

$$\Delta u = A^{-1}(-a\Delta t C u^n - \frac{1}{2}\Delta t^2 a^2 K \tilde{U}^n)$$

Where

$\tilde{U}^n$: Is the solution of the 1st step

Result:

According to the stability analysis, the LF method gives a stable solution when $C \leq 0.57$ and TG3-2S gives a stable solution when $C \leq 0.866$, where "C" is the Courant number whose expression is written below.

$$C = |a| \frac{\Delta t}{h}$$

Where:

h: mesh size

Δt: time step

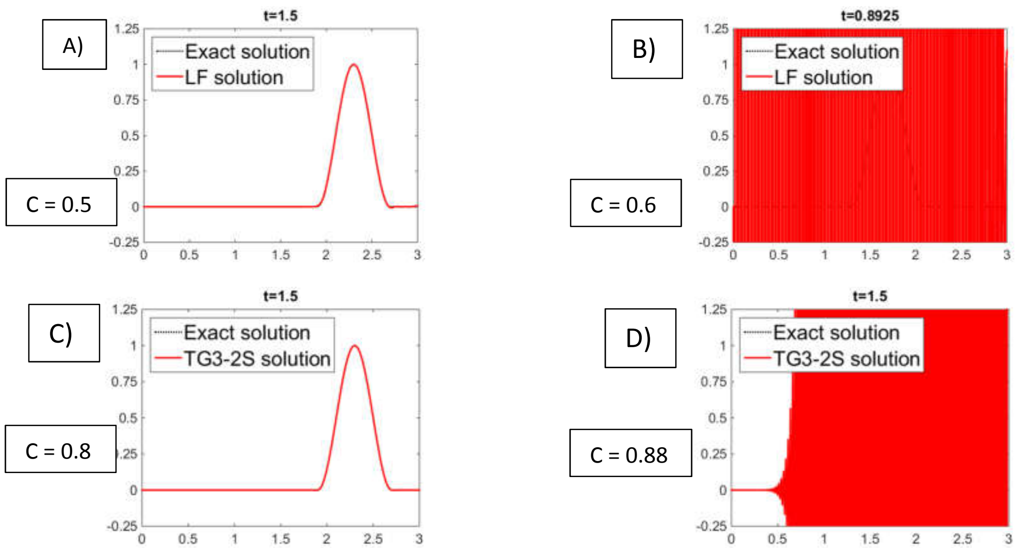

Figure 1: Solution obtained using different methods A) LF C = 0.5 (stable solution), B) LF C = 0.6 (unstable solution), C) TG3-2S C = 0.8 (stable solution), D) TG3-2S C = 0.88. (unstable solution).