FINITE ELEMENTS IN FLUIDS
Marcos Boniquet Aparicio

HOMEWORK 1

1D STEADY CONVECTION-DIFFUSION

$$au_x - \nu u_{xx} = s$$

Marcos Boniquet Aparicio

INTRODUCTION

A steady 1D convection-diffusion domain with Dirichlet boundary conditions is studied with finite elements and different types of methods on stabilization. Four different methods are implemented: **Galerkin, SU, SUPG and GLS.**

The CONVECTION-DIFFUSION equation is as following:

$$au_x - \nu u_{xx} = s \quad x \in [0, 1]$$

and Dirichlet Boundary conditions:

$$U(0) = U_0$$
$$U(1) = U_1$$

Where three different cases are considered at the code:

1. $s = 0$, $u_0 = 0$, $u_1 = 1$
2. $s = 1$, $u_0 = 0$, $u_1 = 0$
3. $s = sin(\pi x)$, $u_0 = 0$, $u_1 = 1$

If we multiply by a test function v (which will be the shape function as for Galerkin), integrating diffusion term by parts, and considering shape function v=0 in $\partial\Omega_D D$ leds to the system:

$$\int_\Omega (v(a \cdot \nabla u) d\Omega + \int_\Omega (\nabla v \cdot (\nu \nabla u) = \int_\Omega vsd\Omega + \int_{\partial\Omega} (vu_N)$$

Because only Dirichlet boundary conditions are applied, the formula is then:

$$\int_\Omega (v(a \cdot \nabla u) d\Omega + \int_\Omega (\nabla v \cdot (\nu \nabla u) = \int_\Omega vsd\Omega$$

this is a linear system $Ku = f$ with:

$$K = \int_\Omega (NaN_x + N_x \nu N_x) d\Omega \ in\ bilinear\ form\ K = a(N, N_x) + c(N_x, N_x)$$

$$f = \int_\Omega Ns\ d\Omega$$

and being N the shape function, equal to the test function.

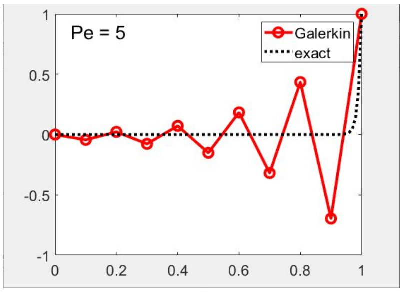
For this homework this linear system is solved for ***a=1,ν=0.01 and 10 linear elements (h=1/10).*** Thus Péclet number is 5, and instabilities are expected to happen if no stabilization method applied.

GALERKIN

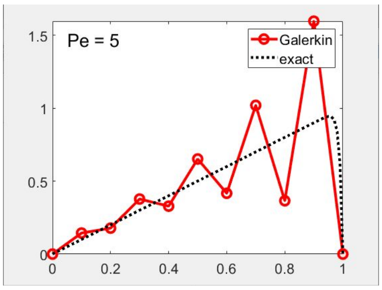
No stabilization is taken. When Péclet number is higher than one $(Pe = \frac{|a|h}{2\nu} > 1)$ oscillations may appear if size of the elements not enough fine. This is because when the approximation for the derivative and second derivative is done the solution form one node to respect for the next is negative:

$$au_x - \nu u_{xx} = s$$
$$a\frac{u_{i+1}-u_{i-1}}{2h} - \nu\frac{u_{i+1}-2u_i+u_{i-1}}{2h} = s$$

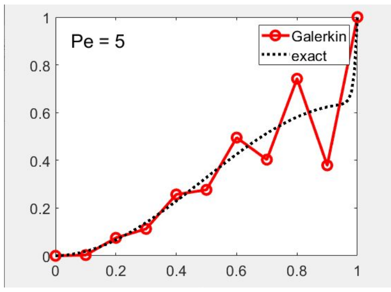
$$u_{i+1} - u_i = \frac{Pe+1}{1-Pe}\left(u_i - u_{i-1}\right)$$



Galerkin for example 1

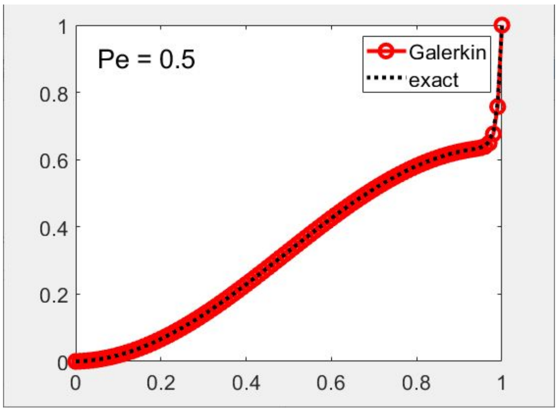


Galerkin for example 2



Galerkin for example 3

For any of the examples the Galerkin method presents instabilities as expected. If mesh is refined to 100 elements for example 3 we shall see how this instabilities also disappears:



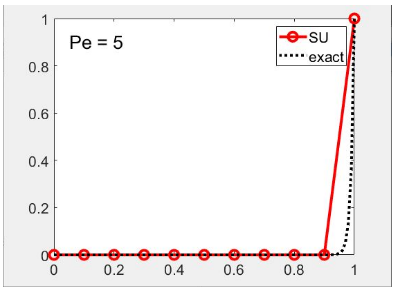
However, this is something we **can not afford** because it increases the computational cost.

SU

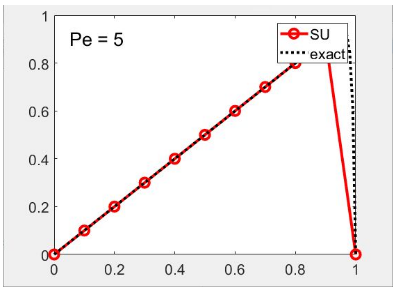
Stabilization term is added. Adding an artificial convective term to erase oscillations just by splitting transport term in upwind and downwind and multiplying them by the term $\frac{1-\beta}{2}$ and $\frac{1+\beta}{2}$ respectively . However the upwind term is given more weight following $\beta = coth Pe - \frac{1}{Pe}$. As a result, when rearranging this splitted equation appears naturally an added diffusion ▿, thus the new diffusion is diffusion plus the added numerical diffusion.

$$K = \int_{\Omega} (N a N_x + N_x(\nu + ▿) N_x) d\Omega$$

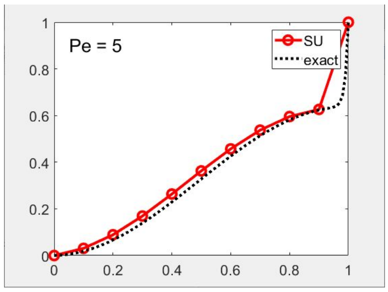
This added diffusion is $▿ = \frac{\beta a h}{2}$, being h the size of the mesh.



SU for example 1



SU for example 2



SU for example 3

Compared to Galerkin, the oscillations disappear, but method is not enough accurate yet unless mesh is refined, which is wanted to be avoid at all cost. Further improvements can be applied in the following stabilization methods.

For the other stabilization methods a different approach is taken.

Aso considering also reaction, it is added an stabilization term which consists of the residual,a stabilization parameter and a term which depends on the method used:

$$\int_{\Omega} (v(a \cdot \nabla u) d\Omega + \int_{\Omega} (\nabla v \cdot (\nu \nabla u) + \sum_{e} \mathcal{P}(v) R(u) d\Omega = \int_{\Omega} v s d\Omega + \int_{\partial\Omega} (v u_N)$$

$$where\ R(u) = (a u_x - \nu\, u_{xx} + \sigma u - s)$$

$\mathcal{P}(v)$ varies depending on method chosen:

- **SUPG** $\mathcal{P}(v) = a \cdot \nabla v$
- **GLS** $\mathcal{P}(v) = a \cdot \nabla v - \nabla \cdot (\nu \nabla v) + \sigma v$ (if no reaction and linear elements is identical to SUPG)

and may be $= \frac{h\beta}{2a}$ (recall: $\beta = coth Pe - \frac{1}{Pe}$) for 1D, same as SU, however for higher order elements it is not clear which one is it.

SUPG

Applying $\mathcal{P}(v) = a \cdot \nabla v$:

Again considering there are only Dirichlet boundary conditions ($\int_{\partial\Omega} (vu_N) = 0$) , the formula is then: $\int_\Omega (v(a \cdot \nabla u)d\Omega + \int_\Omega (\nabla v \cdot (\nu \nabla u) + \sum_e \mathcal{P}(v) = (a \cdot \nabla v)R(u)d\Omega = \int_\Omega vsd\Omega)$

$where\ R(u) = (au_x - \nu\, u_{xx} + \sigma u - s)$, but with linear elements,

Given that no reaction and linear elements are applied, (u_{xx} =0 and σ =0)

$$\int_\Omega (v(a \cdot \nabla u)d\Omega + \int_\Omega (\nabla v \cdot (\nu \nabla u) + \sum_e \mathcal{P}(v) = (a \cdot \nabla v)(au_x - s)d\Omega = \int_\Omega vsd\Omega)$$

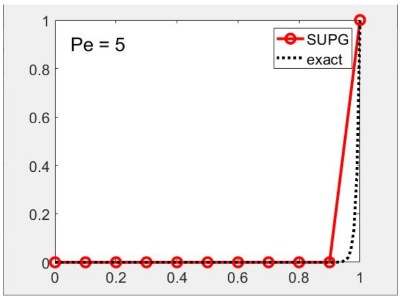
in this particular case without reaction and with linear elements, is remarkable that:

$$K = \int_\Omega (NaN_x + N_x(\nu + \tau a^2)N_x)d\Omega$$

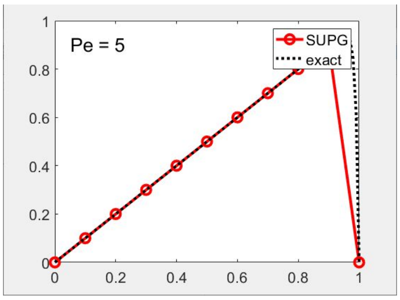
So if term τa^2 is equal to $\tilde{\nu}$ (recall: $\tilde{\nu} = \frac{\beta a h}{2}$,) then SUPG turns out to be SU if s=0. This happens when $= \frac{h\beta}{2a}$. However there is no need to emulate SU, **if another relaxation parameter is introduced it will differ with SU.**

So for this particular exercise the difference between SU and SUPG is the RHS which now is:

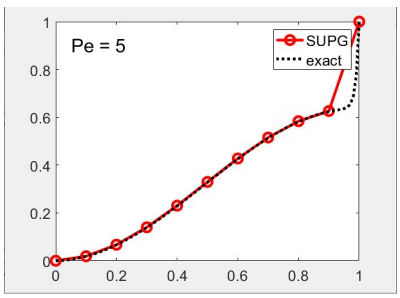
$$f = \int_\Omega (N + \tau a N_x)sd\Omega$$



SUPG for example 1



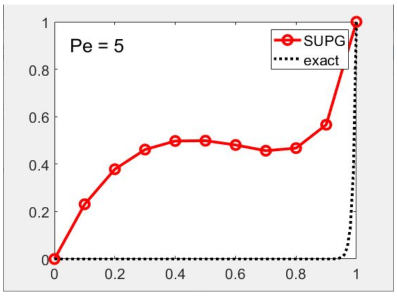
SUPG for example 2



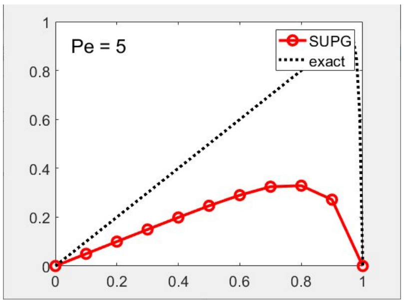
SUPG for example 3

No changes for example 1 given that τa^2 is equal to $\tilde{\nu}$ and s=0. For the example 2 a better approach could be expected, bit being s=ct and different from 0 makes this difference negligible, the SU was good enough. However for the third example the variable source term has impact because it is considered in RHS of the linear system. The curve matches much better the exact solution than SU.

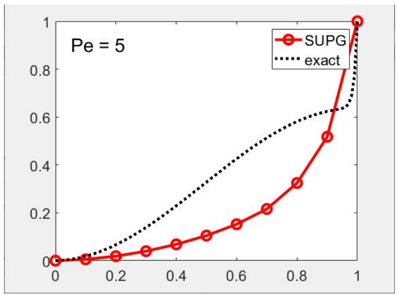
<u>Quadratic SUPG:</u>



SUPG for example 1



SUPG for example 2



SUPG for example 3

Stabilization parameter is probably necessary to be adjusted in order fort the method to approach a better solution.

GLS

For GLS, applying: $\mathcal{P}(v)$ without reaction:

$$\mathcal{P}(v) = a \cdot \nabla v - \nabla \cdot (\nu \nabla v)$$

For no reaction and with no linear elements no differences with SUPG are expected. For his case the reaction is till not considered in order to have the same experiment, however *quadratic elements* are to be included.

$$\int_\Omega (v(a \cdot \nabla u)d\Omega + \int_\Omega (\nabla v \cdot (\nu \nabla u) + \sum_e \mathcal{P}(v) = (a \cdot \nabla v - \nabla \cdot (\nu \nabla v))R(u)d\Omega = \int_\Omega vsd\Omega)$$

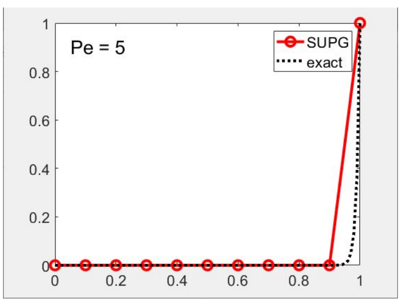
$$where\ R(u) = (au_x - \nu\, u_{xx} - s),\ \text{and again} \ = \frac{h\beta}{2a}$$

The linear system $Ku = f$ now is the following:

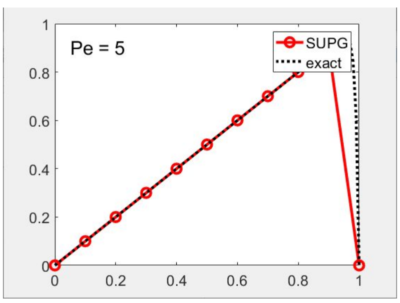
$$K = \int_\Omega (NaN_x + N_x(\nu + \tau a^2)N_x - 2N_{xx}\,(\nu \tau a)N_x + \nu^2 \tau N_{xx} N_{xx})d\Omega$$

$$f = \int_\Omega (N + \tau(aN_x - \nu N_{xx}))sd\Omega$$

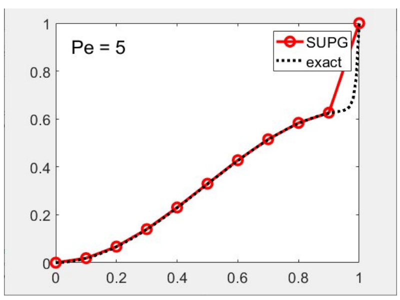
<u>Linear case (no reaction):</u>



GLS linear for example 1



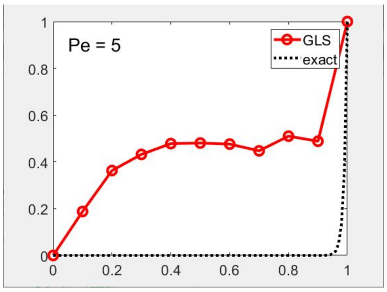
GLS linear for example 2



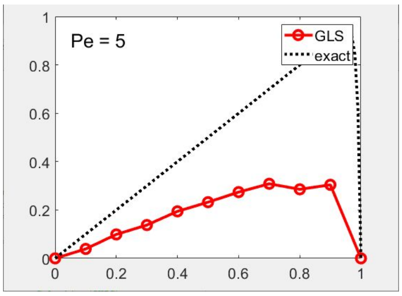
GLS linear for example 3

Remains the same as SUPG.

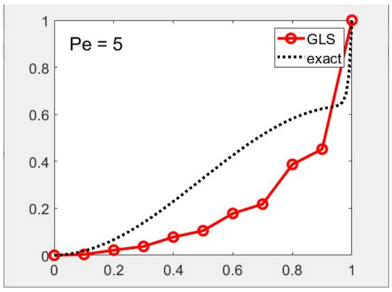
<u>Quadratic case (no reaction):</u>



GLS QUADRATIC for example 1



GLS QUADRATIC for example 2



GLS QUADRATIC for example 3

Stabilization parameter is probably necessary to be adjusted in order fort the method to approach a better solution.

Exponential Source term

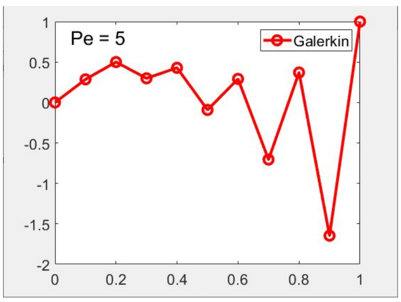
If source term is $s = e^{(-5x)} - 4e^{(-x)}$, with linear elements and same $= \frac{h\beta}{2a}$ then the three examples are compared for each stabilization method:

This leads to:

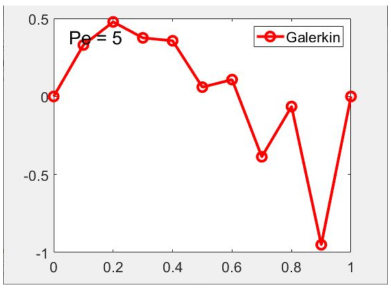
1.b $s = e^{(-5x)} - 4e^{(-x)}$ **,** $u_0 = 0$ **,** $u_1 = 1$ ***with a=1,ν=0.01 and 10 linear elements***
2.b $s = e^{(-5x)} - 4e^{(-x)}$ **,** $u_0 = 0$ **,** $u_1 = 0$ ***with a=1,ν=0.01 and 10 linear elements***
3.b. $s = e^{(-5x)} - 4e^{(-x)}$ **,** $u_0 = 0$ **,** $u_1 = 1$ ***with a=1,ν=0.01 and 10 linear elements***

So 1.b and3.b have same source term and boundary conditions. There is no need to study 3b.

GALERKIN

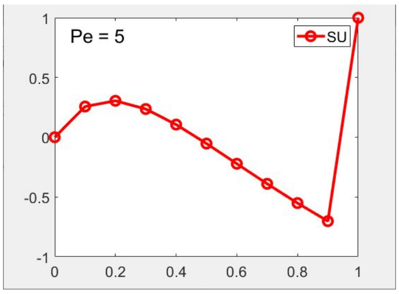


example 1.b

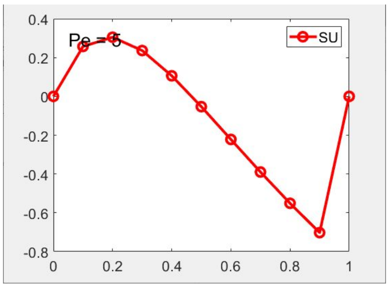


example 2.b

SU

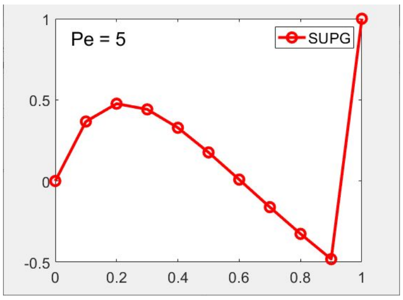


example 1.b

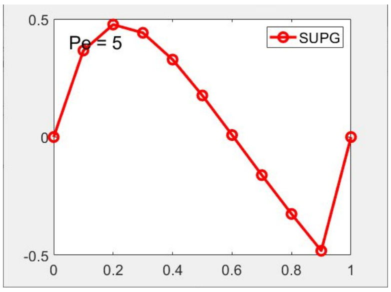


example 2.b

SUPG

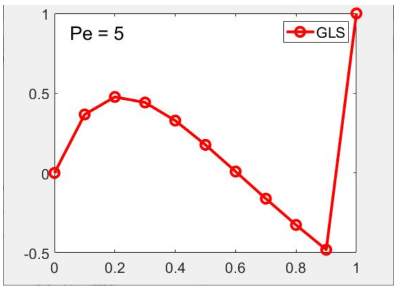


example 1.b

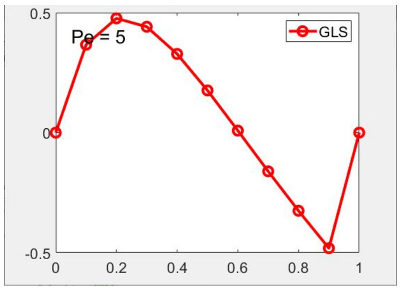


example 2.b

GLS linear and no difussion=SUPG



example 1.b



example 2.b

The exact solution has not been calculated however the approximations with stabilization method for each example present the behavior shown at the slides, with oscillations for the Galerkin solution, a SU without oscillation but not quite matching the exact solucion and a SUPG with an almost exact match along the curve (except for the las element).