Master Of Science in Computational Mechanics

Finite Elements in Fluids

Chinmay Khisti (13 February 2019)

Assignment 1

Convert the conservative form of momentum conservation to non-conservative form

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla\cdot\sigma = \rho\mathbf{b} \rightarrow \frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v} - \sigma) = \rho\mathbf{b} \quad (1)$$

The conservative form of the linear momentum equation is,

$$\frac{\partial \rho\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v} - \sigma) = \rho\mathbf{b}$$
$$\rho\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) - \nabla\cdot\sigma = \rho\mathbf{b}$$

where,

$$\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \rho uu & \rho uv & \rho uz \\ \rho vu & \rho vv & \rho vz \\ \rho wu & \rho wv & \rho ww \end{bmatrix}$$

the x-component of the above equation will give:

$$\frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} + \frac{\partial(\rho wu)}{\partial z}$$
$$= \rho\frac{\partial(uu)}{\partial x} + uu\frac{\partial\rho}{\partial x} + \rho\frac{\partial(vu)}{\partial y} + vu\frac{\partial\rho}{\partial y} + \rho\frac{\partial(wu)}{\partial z} + wu\frac{\partial\rho}{\partial z}$$
$$= \rho\Big(\frac{\partial(uu)}{\partial x} + \frac{\partial(vu)}{\partial y} + \frac{\partial(wu)}{\partial z}\Big) + u\Big(u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial x}\Big)$$
$$= \rho\Big(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} + u\frac{\partial w}{\partial z} + w\frac{\partial u}{\partial z}\Big) + u\begin{bmatrix} u & v & w \end{bmatrix}\cdot\begin{bmatrix} \frac{\partial\rho}{\partial x} & \frac{\partial\rho}{\partial y} & \frac{\partial\rho}{\partial z} \end{bmatrix}$$
$$= \rho\Big(u\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big) + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\Big) + u(\mathbf{v}\cdot\nabla\rho)$$
$$= \rho\Big(u(\nabla\cdot\mathbf{v}) + \mathbf{v}\cdot\nabla u\Big) + u(\mathbf{v}\cdot\nabla\rho)$$

similarly we can find y and z-components of the expression. Thus, it satisfy the equation

$$\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) = \rho\Big(\mathbf{v}(\nabla\cdot\mathbf{v}) + \mathbf{v}\cdot\nabla\mathbf{v}\Big) + \mathbf{v}(\mathbf{v}\cdot\nabla\rho)$$

expanding the right hand side of the equation

$$\rho\Big(\mathbf{v}(\nabla\cdot\mathbf{v}) + \mathbf{v}\cdot\nabla\mathbf{v}\Big) + \mathbf{v}\Big(\nabla\cdot(\rho\mathbf{v}) - \rho(\nabla\cdot\mathbf{v})\Big)$$
$$= \rho\mathbf{v}(\nabla\cdot\mathbf{v}) + \rho\mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{v}\nabla\cdot(\rho\mathbf{v}) - \rho\mathbf{v}(\nabla\cdot\mathbf{v})$$
$$= \rho\mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{v}\nabla\cdot(\rho\mathbf{v})$$

Substituting in equation (1):

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial\rho}{\partial t} + \rho\mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{v}\nabla\cdot(\rho\mathbf{v}) - \nabla\cdot\sigma = \rho\mathbf{b}$$
$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \mathbf{v}\Big(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v})\Big) - \nabla\cdot\sigma = \rho\mathbf{b}$$
$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla\cdot\sigma = \rho\mathbf{b}$$