1 Prove the identity of the conservative and non-conservative forms of the momentum conservation equation

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla\cdot\sigma = \rho\mathbf{b} \leftrightarrow \frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v} - \sigma) = \rho\mathbf{b} \tag{1}$$

Starting from the conservative form of the linear momentum equation, it is possible to derive the following:

$$\frac{\partial\rho\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v} - \sigma) = \rho\mathbf{b} \tag{2}$$

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) - \nabla\cdot\sigma = \rho\mathbf{b} \tag{3}$$

The third term of the equation (3) can be analyzed as it follows:

$$\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) = \begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}\rho uu & \rho uv & \rho uz\\ \rho vu & \rho vv & \rho vz\\ \rho wu & \rho wv & \rho ww\end{bmatrix} \tag{4}$$

We can find the x-component as:

$$\frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} + \frac{\partial(\rho wu)}{\partial z}$$

$$\rho\Big(\frac{\partial(uu)}{\partial x} + \frac{\partial(vu)}{\partial y} + \frac{\partial(wu)}{\partial z}\Big) + u\Big(u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial x}\Big)$$

$$\rho\Big(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} + u\frac{\partial w}{\partial z} + w\frac{\partial u}{\partial z}\Big) + u\begin{bmatrix}u & v & w\end{bmatrix}\cdot\begin{bmatrix}\frac{\partial\rho}{\partial x} & \frac{\partial\rho}{\partial y} & \frac{\partial\rho}{\partial z}\end{bmatrix}$$

$$\rho\Big(u\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big) + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\Big) + u(\mathbf{v}\cdot\nabla\rho)$$

$$\rho\Big(u\nabla\cdot\mathbf{v} + \begin{bmatrix}u & v & w\end{bmatrix}\cdot\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\end{bmatrix}\Big) + u(\mathbf{v}\cdot\nabla\rho)$$

$$\rho\Big(u(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla u\Big) + u(\mathbf{v} \cdot \nabla \rho)$$

And analogous the y-component and z-component. The following identity is satisfied:

$$\nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) = \rho\Big(\mathbf{v}(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla \mathbf{v}\Big) + \mathbf{v}(\mathbf{v} \cdot \nabla \rho) \tag{5}$$

Using the chain rule:

$$\rho\Big(\mathbf{v}(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla \mathbf{v}\Big) + \mathbf{v}\Big(\nabla \cdot (\rho \mathbf{v}) - \rho(\nabla \cdot \mathbf{v})\Big)$$

$$\rho \mathbf{v}(\nabla \cdot \mathbf{v}) + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v} \nabla \cdot (\rho \mathbf{v}) - \rho \mathbf{v}(\nabla \cdot \mathbf{v})$$

$$\rho \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v} \nabla \cdot (\rho \mathbf{v})$$

Substituting into the original equation:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v} \nabla \cdot (\rho \mathbf{v}) - \nabla \cdot \sigma = \rho \mathbf{b}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} + \mathbf{v}\Big(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\Big) - \nabla \cdot \sigma = \rho \mathbf{b}$$

$$\boxed{\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla \cdot \sigma = \rho \mathbf{b}}$$