Finite Element in Fluids

Incompressible Flow

Stokes Equation

<u>Home Work -7</u>

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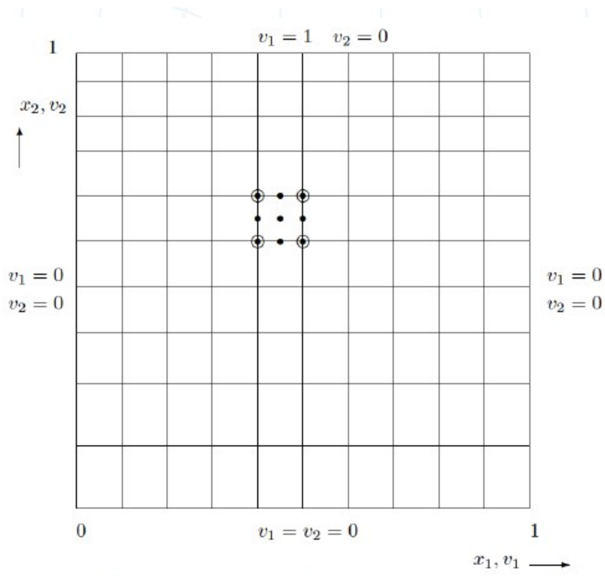
MS-Computational Mechanics

<u>Problem</u>

Stokes Equation

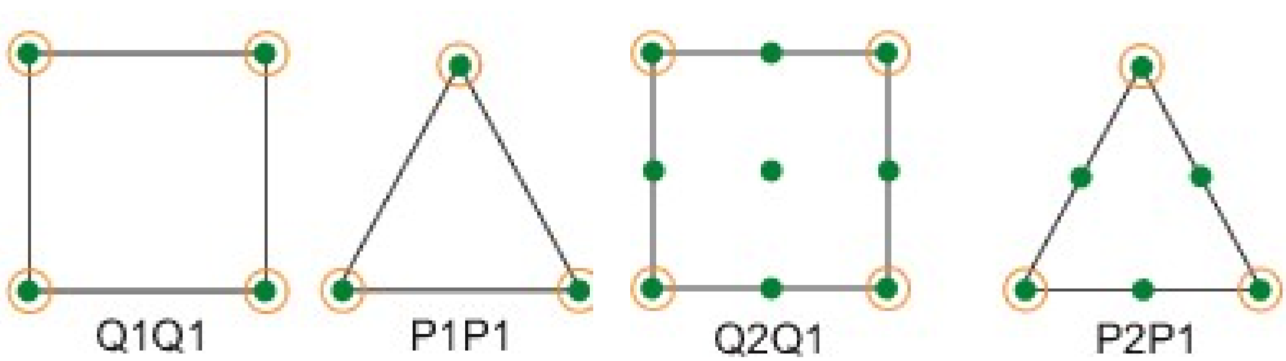
$$\begin{cases} -\nu \nabla^2 \boldsymbol{v} + \boldsymbol{\nabla} p = \boldsymbol{f} & \text{in } \Omega \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 & \text{in } \Omega \\ \boldsymbol{v} = \boldsymbol{v}_D & \text{on } \partial\Omega \end{cases}$$

Physical Description of the Problem

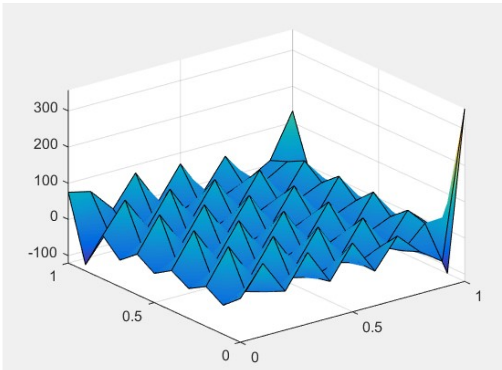


The two dimensional problem in the square domain Ω =]0, 1[x]0, 1[, with boundary conditions can be seen in above diagram. It poses a close solution with the velocity field v = (v1,v2) and pressure field p.

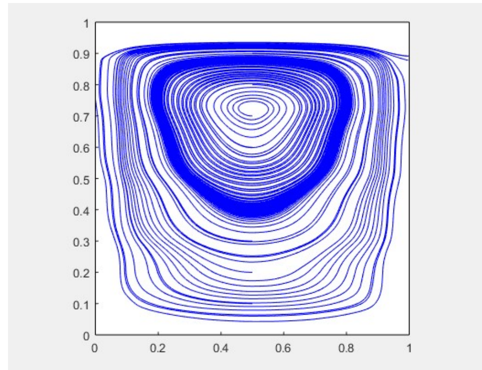
The problem is descritized with 10 elements in each direction. Four different types of elements have been chosen to analysis the problem. Q1Q1, P1P1, Q2Q1 & P2P1.



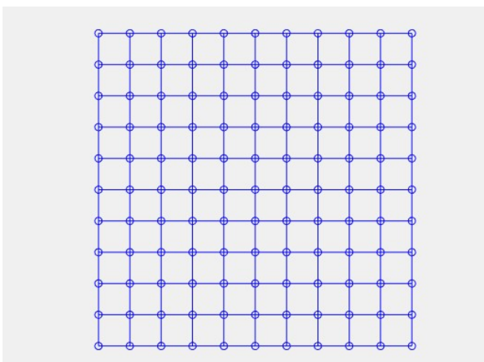
a) Q1Q1 Element



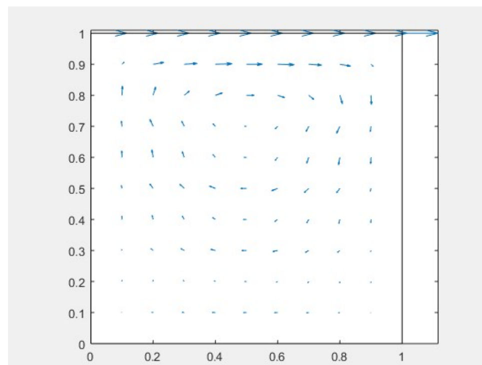
Pressure Field



Streamlines



Elements scheme

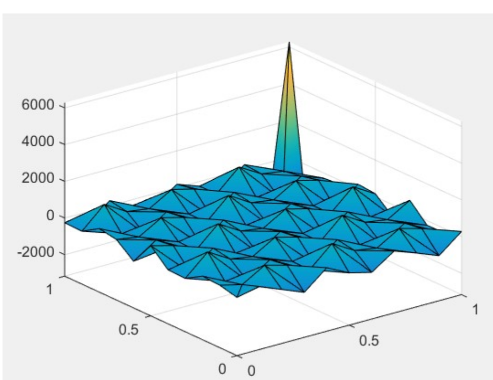


Velocity vectors

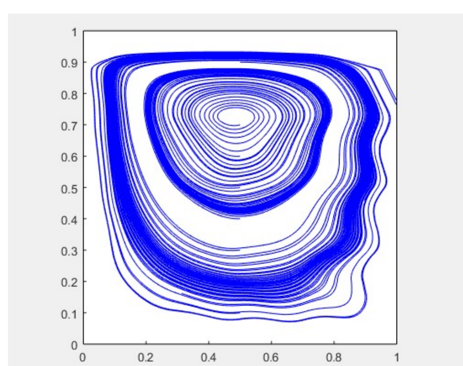
Figure-1: Response of Q1Q1 element

It can be seen that continues bilinear velocity and continues bilinear pressure element does not satisfy the LBB compatibility condition. And as per expectation it shows in-accuracy in term of pressure and solution oscillates from element to element and it is dominant across the field.

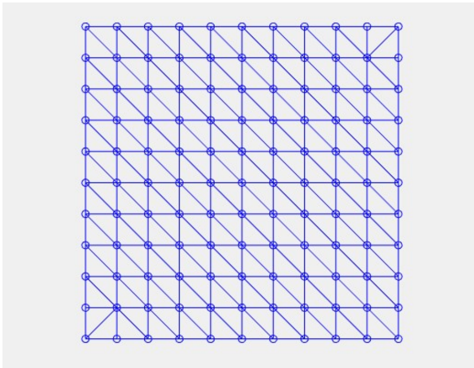
b) P1P1 Element



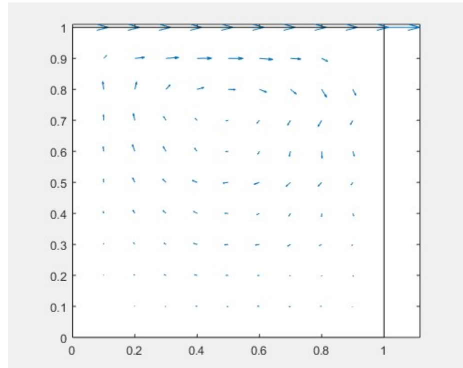
Pressure Field



Streamlines



Elements scheme

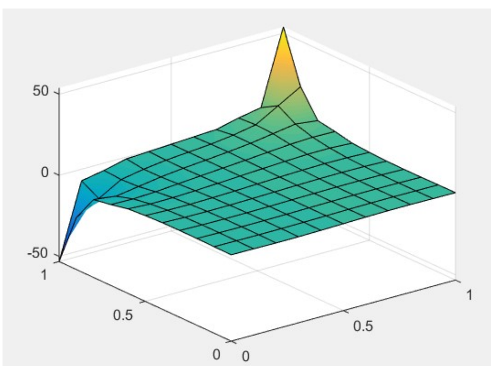


Velocity vectors

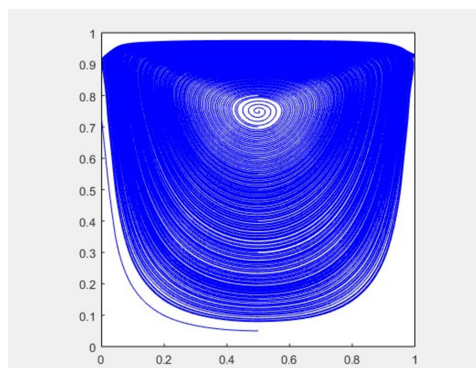
Figure-2: Response of P1P1 element

It can be seen that continues triangular elements (velocity & pressure) does not satisfy the LBB compatibility condition. And response is highly inaccurate in terms of pressure field. Even it has larger element to element oscillation at corners and streamlines shows discontinuity.

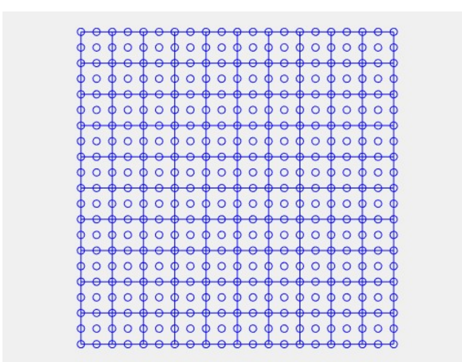
c) Q2Q1 Element



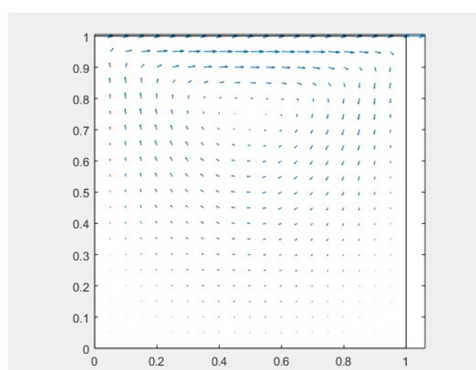
Pressure Field



Streamlines



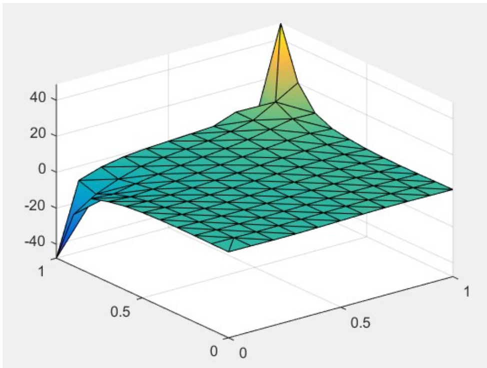
Elements scheme



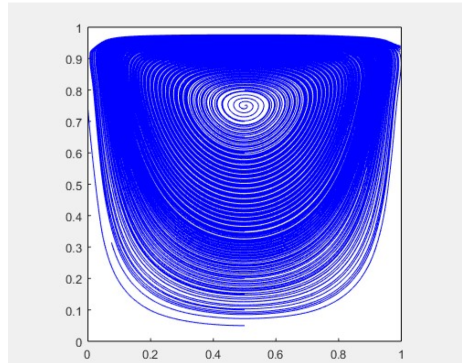
Velocity vectors

Figure-3: Response of Q2Q1 element

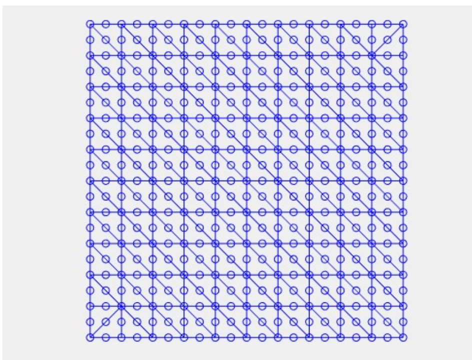
d) P2P1 Element



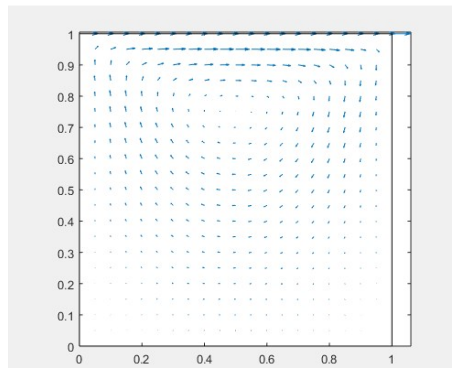
Pressure Field



Streamlines



Elements scheme



Velocity vectors

Figure-4: Response of P2P1 element

Both Q2Q1 & P2P1 elements give stable finite solutions. Pressure field has a smooth trend across the entire field while velocity streamlines are very smooth and continues.

Stabilization for Q1Q1 and P1P1 elements

The matlab code for stokes equation has been modified using GLS stabilized formulation Q1Q1 & P1P1 to overcome the element to element oscillations. The stabilized weak form of the equation is as below

$$a(\boldsymbol{w}^h, \boldsymbol{v}^h) + b(\boldsymbol{w}^h, p^h) = (\boldsymbol{w}^h, \boldsymbol{b}^h) + (\boldsymbol{w}^h, \boldsymbol{t}^h)_{\Gamma_N},$$

$$b(\boldsymbol{v}^h, q^h) - \sum_{e=1}^{n_{el}} \tau_e (\boldsymbol{\nabla} q^h, \boldsymbol{\nabla} p^h)_{\Omega^e} = -\sum_{e=1}^{n_{el}} \tau_e (\boldsymbol{\nabla} q^h, \boldsymbol{b}^h)_{\Omega^e}$$

In the weak form of the stokes equation, the stabilized term is incorporated. As linear element is used, the GLS stabilization does not affect the weak form of the momentum equation because the terms involving the second derivatives of the weighting function vanishes. The stabilization parameter used is:

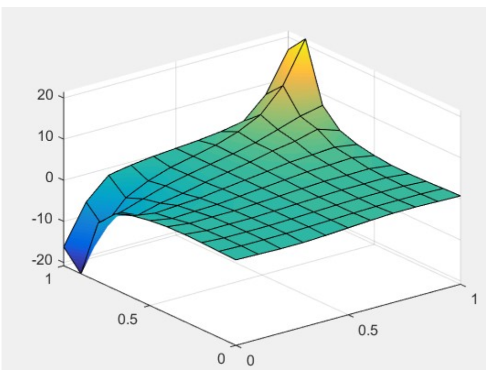
$$\tau_e = \alpha_0 \frac{h_e^2}{4\nu},$$

$$\alpha_0 = 1/3$$

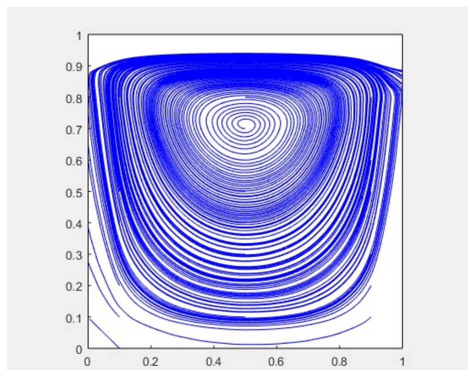
$$\nu = 1, h = 0.1$$

The result of the stabilization is as follows

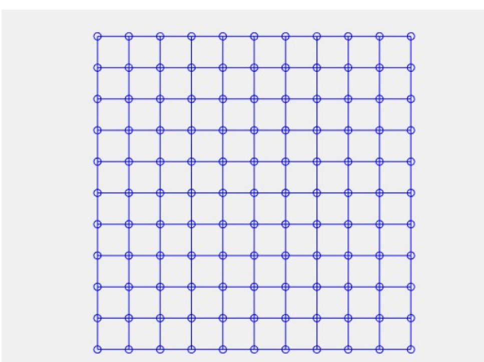
e) Stabilized Q1Q1 Element



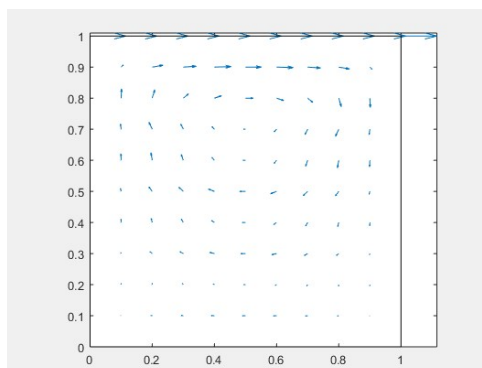
Pressure Field



Streamlines



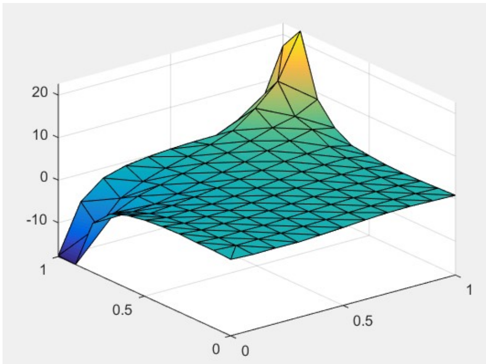
Elements scheme



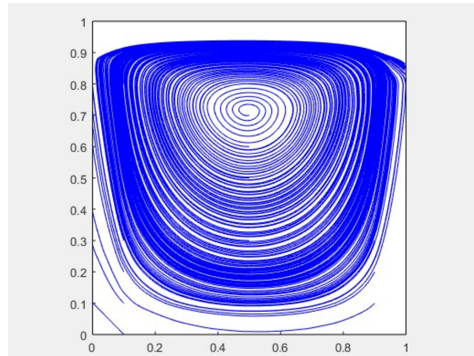
Velocity vectors

Figure-5: Response of stabilized Q1Q1 element

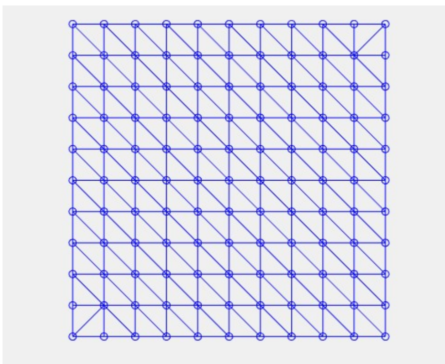
f) Stabilized P1P1 Element



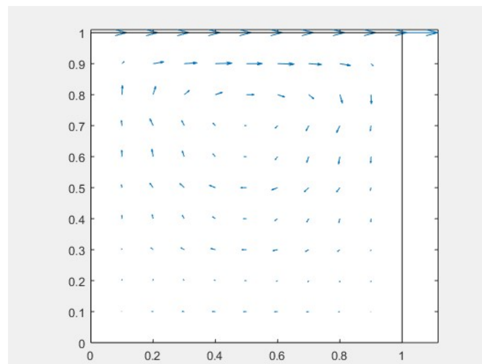
Pressure Field



Streamlines



Elements scheme



Velocity vectors

Figure-6: Response of stabilized P1P1 element

It can be seen that pressure field is smooth and all the oscillations are vanished and GLS stabilized scheme for both Q1Q1 & P1P1 produce stable solution of stokes equation.