Finite Element in Fluids

HDG

Assignment. No. 15

Home Work -10

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MS-Computational Mechanics

Problem Statement

FINITE ELEMENTS IN FLUIDS

HDG assignment #15

INSTRUCTIONS

- *Read carefully the questions and answer in a* pertinent, clear *and* concise *way.*
- *Report due by June 5, 2019.*

Consider the domain $\Omega = [0,1]^2$ such that $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R$ with $\Gamma_D \cap \Gamma_N = \emptyset$, $\Gamma_D \cap \Gamma_R = \emptyset$ and $\Gamma_N \cap \Gamma_R = \emptyset$. More precisely, set

$$\Gamma_N := \left\{(x,y) \in \mathbb{R}^2 \ : \ y = 1\right\},$$
$$\Gamma_R := \left\{(x,y) \in \mathbb{R}^2 \ : \ x = 1\right\},$$
$$\Gamma_D := \partial\Omega \setminus (\Gamma_N \cup \Gamma_R).$$

The following second-order linear scalar partial differential equation is defined

$-\nabla . \nabla u = f$

$u = u_D$

$n . \nabla u = t$

$$\begin{cases} -\nabla\cdot(\kappa\nabla u) = s & \text{in } \Omega, \\ u = u_D & \text{on } \Gamma_D, \\ \boldsymbol{n}\cdot(\kappa\nabla u) = t & \text{on } \Gamma_N, \\ \boldsymbol{n}\cdot(\kappa\nabla u) + \gamma u = g & \text{on } \Gamma_R, \end{cases} \tag{P}$$

where κ and γ are the diffusion and convection coefficients, respectively, $\boldsymbol{n}$ is the outward unit normal vector to the boundary, s is a volumetric source term and u_D, t and g are the Dirichlet, Neumann and Robin data imposed on the corresponding portions of the boundary $\partial\Omega$.

1. Write the HDG formulation of the problem (P). More precisely, derive the HDG strong and weak forms of the local and global problems. [*Hint: the hybrid variable $\hat{u}$ needs to be introduced both on Γ_N and Γ_R.*]
2. Implement in the Matlab code provided in class the corresponding HDG solver.
3. Set κ=6 and γ=1.6. Consider $u(x,y)=\cos(\cos(\kappa\pi y) + a\exp(\gamma x + by))$, with a=−1.2 and b=0.9. Determine the analytical expressions of the data u_D, t and g in problem (P). [*Hint: use Matlab tools for symbolic calculus.*]
4. Solve problem (P) using HDG with different meshes and polynomial degrees of approximation. Starting from the plots provided by the Matlab code, discuss the accuracy of the obtained solution u and of the postprocessed one $u^\star$.
5. Compute the errors for u, $\boldsymbol{q}$ and $u^\star$ in the $\mathcal{L}_2$-norm defined on the domain Ω. Perform a convergence study for the primal, u, mixed, $\boldsymbol{q}$ and postprocessed, $u^\star$ variables for a polynomial degree of approximation k=1, . . . , 4. Discuss the obtained numerical results, starting from the theoretical results on the optimal convergence rates of HDG.

1. HDG Strong & Weak forms of Local & Global problem

The strong form of the problem (P) is written within the broken computational domain.

Strong form in Broken Computational Domain

Robin boundary Condition is applied together with Dirichlet & Neumann boundary Conditions.

$$-\nabla\cdot(K\nabla u) = s \quad \text{in } \Omega_i, \text{ and for } i=1,\dots,n_{el}$$
$$u = u_D \quad \text{on } \Gamma_D$$
$$n\cdot(K\nabla u) = t \quad \text{on } \Gamma_N$$
$$n\cdot(K\nabla u) + \gamma u = g \quad \text{on } \Gamma_R$$
$$[\![un]\!] = 0 \quad \text{on } \Gamma$$
$$[\![n\cdot\nabla u]\!] = 0 \quad \text{on } \Gamma$$

Local Problem

local problem with dirichlet boundary condition is defined as

$$\nabla\cdot q_i = s \quad \text{in } \Omega_i$$
$$q_i + K\nabla u_i = 0 \quad \text{in } \Omega_i$$
$$u_i = u_D \quad \text{on } \partial\Omega_i \cap \Gamma_D$$
$$u_i = \hat{u} \quad \text{on } \partial\Omega_i \setminus \Gamma_D$$

For $i = 1, \dots, n_{el}$

Global Problem

Global problem is defined to determined $\hat{u}$ where $\hat{u}$ is imposed on both Γ_N & Γ_R.

$$[\![n\cdot q]\!] = 0 \quad \text{on } \Gamma$$
$$n\cdot q = -t \quad \text{on } \Gamma_N$$
$$n\cdot q - \gamma\hat{u} = -g \quad \text{on } \Gamma_R$$

The equation is automatically satisfied as $u = \hat{u}$ on Γ as $[\![un]\!] = 0$, Condition satisfied becaus $\hat{u}$ is unique for adjacent element.

Weak Form of Local Problem

Weak formulation for each element is given as follow

$$-(\nabla v, q_i)_{\Omega_i} + \langle v, n_i \cdot \hat{q}_i \rangle_{\partial\Omega_i} = (v, s)_{\Omega_i}$$

$$-(w, q_i)_{\Omega_i} + K(\nabla \cdot w, u_i)_{\Omega_i} = K\langle n_i \cdot w, u_D \rangle_{\partial\Omega_i \cap \Gamma_D} + K\langle n_i \cdot w, \hat{u} \rangle_{\partial\Omega_i \setminus \Gamma_D}$$

After integrating by parts and Substituting numerical fluxes with

$$n_i \cdot \hat{q}_i = \begin{cases} n_i \cdot q_i + \tau_i (u_i - u_D) & \text{on } \Omega_i \cap \Gamma_D \\ n_i \cdot q_i + \tau_i (u_i - \hat{u}), & \text{elsewhere} \end{cases}$$

Weak Form of Global Problem

By adding the Robin boundary Condition

$$\sum_{e=1}^{ne1} \langle \mu, n_i \cdot \hat{q} \rangle_{\partial\Omega_i \setminus \partial\Omega} + \sum_{e=1}^{ne1} \langle \mu, n_i \cdot \hat{q} + t \rangle_{\partial\Omega_i \cap \Gamma_N} + \sum_{e=1}^{ne1} \langle \mu, n_i \cdot \hat{q} - \gamma \hat{u} + g \rangle_{\partial\Omega_i \cap \Gamma_R} = 0$$

Replacing the defination of fluxes

$$\sum_{m=1}^{ne1} \left\{ \langle \mu, n_i \cdot q_i \rangle_{\partial\Omega_i \setminus \Gamma_D} + \langle \mu, \tau_i u_i \rangle_{\partial\Omega_i \setminus \Gamma_D} - \langle \mu, \tau_i \hat{u} \rangle_{\partial\Omega_i \setminus \Gamma_D} - \langle \mu, \gamma \hat{u} \rangle_{\partial\Omega_i \cap \Gamma_R} \right\} = -\sum_{m=1}^{ne1} \left\{ \langle \mu, g \rangle_{\partial\Omega_i \cap \Gamma_R} + \langle \mu, t \rangle_{\partial\Omega_i \cap \Gamma_N} \right\}$$

After discretization of local Equation, we have

$$\begin{bmatrix} A_{uu} & A_{uq} \\ KA_{uq}^T & A_{qq} \end{bmatrix}_i \begin{bmatrix} u_i \\ q_i \end{bmatrix} = \begin{bmatrix} f_u \\ Kf_q \end{bmatrix} + \begin{bmatrix} A_{u\hat{u}} \\ KA_{q\hat{u}} \end{bmatrix}_i \hat{u}$$

After discretization of global

$$\sum_{n=1}^{nel} \left\{ \begin{bmatrix} A_{u\hat{u}}^T & A_{q\hat{u}}^T \end{bmatrix} \begin{bmatrix} u_i \\ q_i \end{bmatrix} + [A_{\hat{u}\hat{u}}]_i \hat{u}_i + [A_{\hat{u}\hat{u}}^R] \hat{u}_i \right\} = \sum_{n=1}^{nel} \left\{ [f_{\hat{u}}]_i + [f_{\hat{u}}^R]_i \right\}$$

where matrix $A_{\hat{u}\hat{u}}^R$ and $f_{\hat{u}}^R$ are associated to Robin boundry condition of the condition

$$A_{\hat{u}\hat{u}}^R = - \sum_{\partial\Omega_i \cap \Gamma_R} \gamma \sum_{g=1}^{n_{ip}^f} \hat{N}_n(\xi_g^f) \hat{N}^T(\xi_g^f) w_g^f$$

$$f_{\hat{u}}^R = - \sum_{\partial\Omega_i \cap \Gamma_R} \sum_{g=1}^{n_{ip}^f} N(\xi_g^f) g(x(\xi_g^f)) w_g^f$$

The global system

$$\hat{K}\hat{u} = \hat{f}$$

$$\hat{K} = \Pi_{i=1}^{nel} \begin{bmatrix} A_{u\hat{u}}^T & A_{q\hat{u}}^T \end{bmatrix} \begin{bmatrix} A_{uu} & A_{uq} \\ KA_{uq}^T & A_{qq} \end{bmatrix}_i^{-1} \begin{bmatrix} A_{u\hat{u}} \\ KA_{q\hat{u}} \end{bmatrix}_i$$

$$+ [A_{\hat{u}\hat{u}}]_i + [A_{\hat{u}\hat{u}}^R]_i$$

and

$$\hat{f} = \Pi_{i=1}^{nel} [f_{\hat{u}}]_i + [f_{\hat{u}}^R]_i - \begin{bmatrix} A_{u\hat{u}}^T & A_{q\hat{u}}^T \end{bmatrix}_i \begin{bmatrix} A_{uu} & A_{uq} \\ KA_{uq}^T & A_{qq} \end{bmatrix}_i^{-1} \begin{bmatrix} f_u \\ Kf_q \end{bmatrix}_i$$

2. MatLab Code Implementation

A few changes are made to MatLab code in order to implement HDG method to work for the flow problem. First step is to separate different boundary conditions of Neumann and Robin because code is for solving pure dirichlet boundary conditions. So, exterior faces are created for dirichlet, neumann & robin boundary conditions. Modifications are done in GetFaces.m and Preprocess.m.

DoF of unknowns are added to mainPoissonal.HDG.m while in hdg_MatrixPoisson.m, the global system of equations is solved.

In HDGPostprocess.m, viscosity is added and Matlab symbolic calculus tools are used to determine the analytical expression of the problem given.

3. Analytical Expression

The analytical expression for ***s***, ***u_D***, ***t*** and ***g*** for the given data

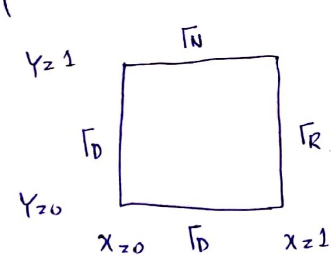
Set κ=6 and γ=1.6. Consider $u(x,y) = \cos(\cos(\kappa\pi y) + a\exp(\gamma x + by))$, with a=−1.2 b=0.9.

can be calculated by following Matlab code.

```
syms a b x y kappa gamma

u = cos(cos(kappa*pi*y) + a*exp(gamma*x + b*y));

ddx = diff(u,x);

ddy = diff(u,y);

q = -kappa*(ddx+ddy);
%q1 = -kappa*[ddx;ddy];
s = -kappa*(diff(ddx,x) + diff(ddy,y));
%s1 = -kappa*[diff(ddx,x);diff(ddy,y)]
t = -kappa*ddy;
g = kappa*ddx + gamma*u;
```

The expressions are given below

As Boundary Conditions are



As u_D is same as analytical expression.

So Dirichlet

$$u_D = \begin{cases} \cos(\cos(k\pi y) + a.\exp(by)) & \boxed{\text{at } x = 0} \\ \cos(1 + a\exp(\gamma x)) & \boxed{\text{at } y = 0} \end{cases}$$

Neumann

$$t = -k\{\sin(\cos(k\pi) + a.\exp(\gamma x + b)) * \cdots$$
$$* (k\pi\sin(k\pi) - ab.\exp(\gamma x + b))\} \quad \boxed{\text{at } y = 1}$$

Robin

$$g = \gamma\cos(\cos(k\pi y) + a.\exp(\gamma + by))$$
$$- ak\gamma.\exp(\gamma + by)\sin(\cos(k\pi y) + a.\exp(\gamma + by)) \quad \boxed{\text{at } x = 1}$$

$$S = k\Big[\sin(\cos(k\pi y) + a.\exp(\gamma x + by)) * (ab^2.\exp(\gamma x + by) - k^2\pi\cos(k\pi y))$$
$$+ \cos(\cos(k\pi y) + a.\exp(\gamma x + by)) * (k\pi\sin(k\pi y) - ab.\exp(\gamma x + by))^2$$
$$+ a\gamma^2.\exp(\gamma x + by)\sin(\cos(k\pi y) + a.\exp(\gamma x + by))$$
$$+ a^2\gamma^2\exp(2\gamma x + 2by)\cos(\cos(k\pi y) + a.\exp(\gamma x + by))\Big]$$

4. Results and Discussion

The results are presented below for both *u* and *u** for same mesh for different degree of polynomial and it can be seen that *u** gives more accurate results than *u* for same degree of polynomial. While the error decreases as we increase the degree of polynomial.

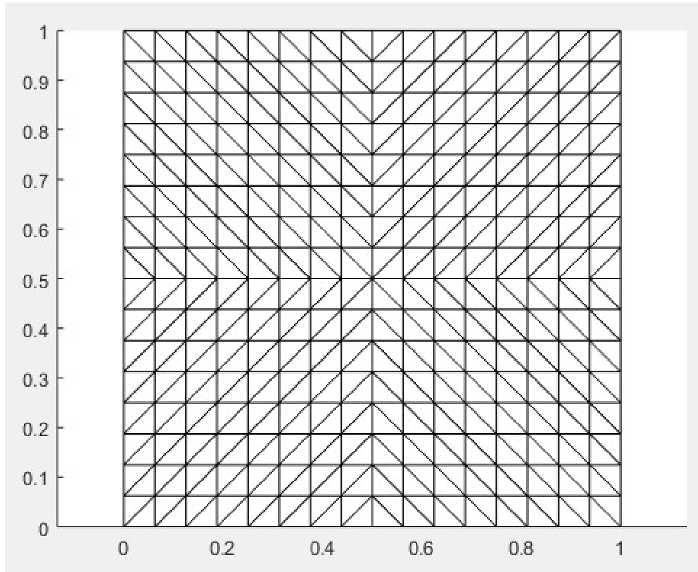


Figure-1: Meshed domain

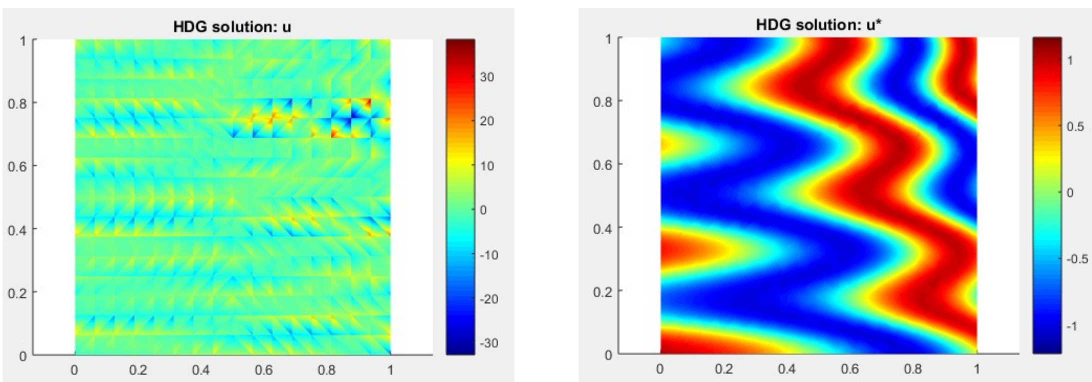


Figure-2: Polynomial of degree 1

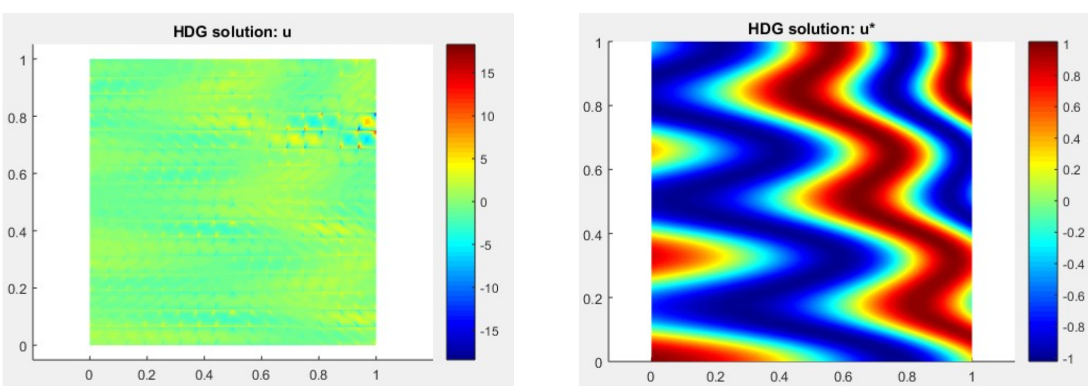


Figure-3: Polynomial of degree 2

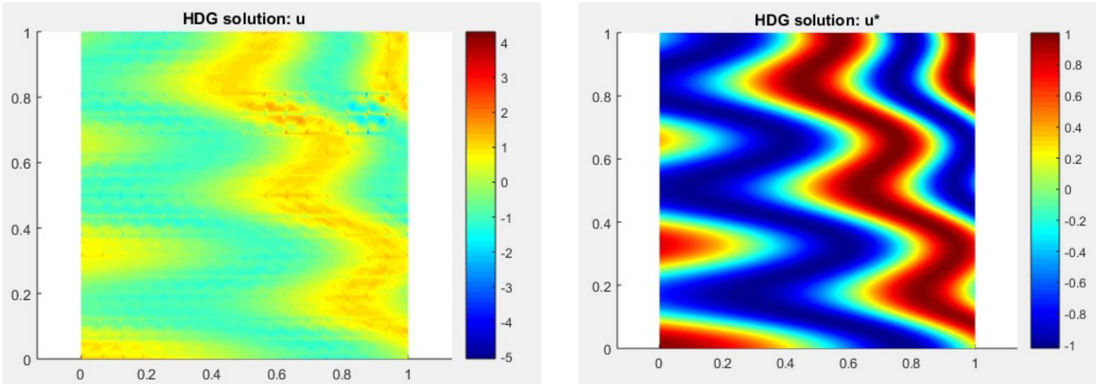


Figure-4: Polynomial of degree 3

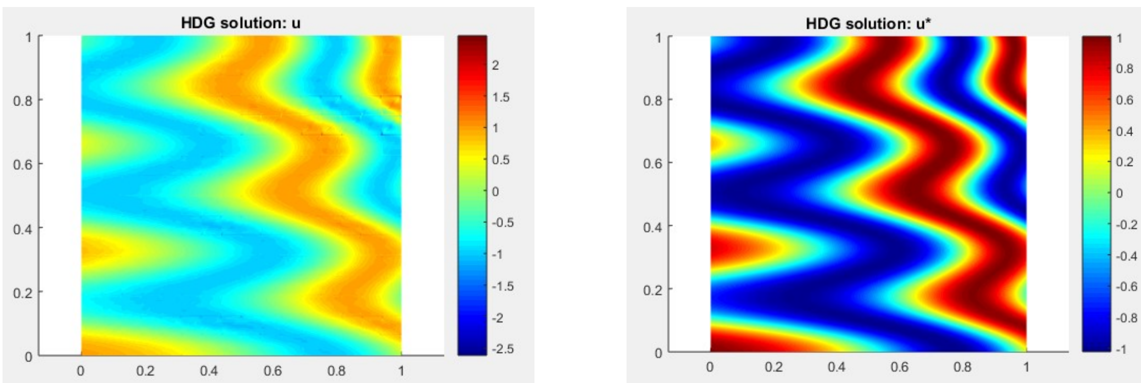


Figure-5: Polynomial of degree 4

L2 Error of the variable		
Degree of Polynomial	$\| u \ u^h \|$	$\| u^* - u^{h*} \|$
1	3.480699e+00	1.597759e-02
2	7.874257e-01	1.006809e-02
3	1.489353e-01	9.872827e-03
4	3.070532e-02	9.868543e-03

Table-1 comparison of Errors