# Non-linear hyperbolic problems

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## Introduction

In many problems in physics the differential equations to be solved are non-linear. This fact adds an extra difficulty when solving numerically the problem. One example of that is the Burger's equation which rules phenomena such as fluid mechanics, non-linear acoustics, gas dynamics, traffic flow… In this study Burger's equation will be solved with different methods.

## Problem Statement

The problem to be solved is the 1D Burger's equation which reads:

$$u_t + u\, u_x = 0$$

setting the initial configuration $u_0$=u(t=0) the evolution of the solution is obtained with the governing equation.
In order to solve this problem, three different methods will be used. An explicit method, which allows us to obtain the solution at the next time step directly with the information of the previous step, and two implicit methods which requires to solve a non-linear system of equations finding, iteratively, the 'root' of function using Picard and Newton-Rapson method.

## Results

In the following section the results obtained for different methods are presented and commented the differences between them. The problem to be solved is the stated in the previous section with initial configuration as shown in *Figure 1.*

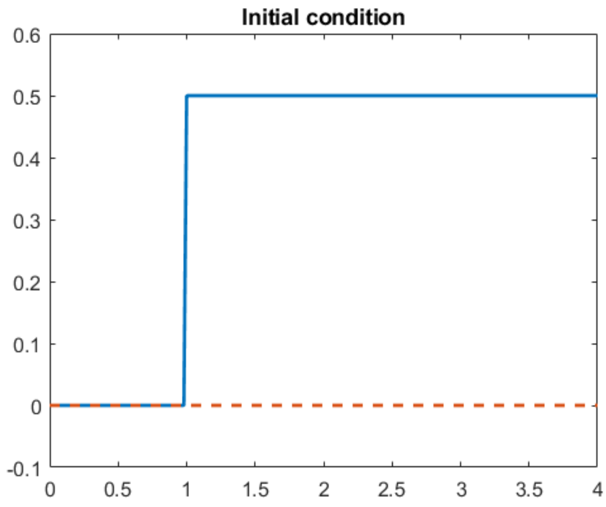


**Figure 1.** Initial configuration of our solution.

With this initial configuration and the governing equation, taking the following parameters: $t_f$=4s, $\Delta$t=0.005s and numerical diffusion $\varepsilon=10^{-2}$ the solution for all methods is:

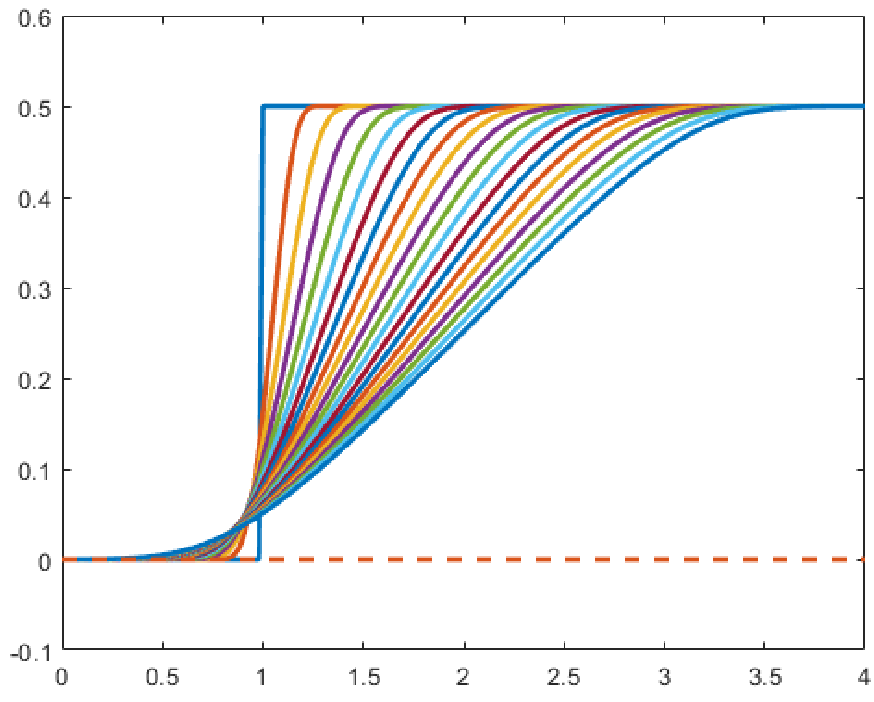


**Figure 2.** Evolution of the solution for Euler, implicit Picard and Newton-Rapson each 0.1 seconds.

In this case, It must be remarked that the solution is pretty much the same for all methods. That is the reason why only one plot has been presented.

When numerical diffusion is not high enough, for example, if instead of $\varepsilon=10^{-2}$, we have $\varepsilon=10^{-3}$, which is lower than before, the following results are obtained:

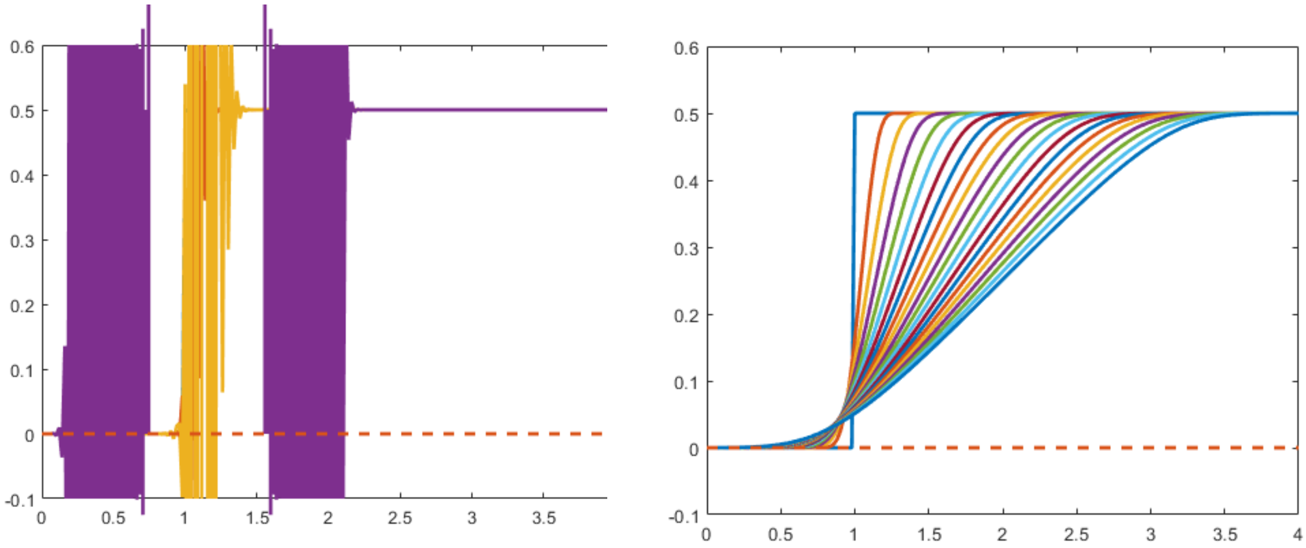


**Figure 3.** Numerical solution for $\Delta t=0.05$ and with numerical diffusion $\varepsilon=10^{-2}$.
For explicit method (left) and implicit methods (right).

As it can be shown, when the diffusion parameter is not high enough, the solution for the explicit method 'blows up' and the solution can't be found. However, for the implicit methods the solution is still accurate.

It has been seen that numerical diffusion affects the behaviour of the method, however, not only $\varepsilon$ affects the solution, also $\Delta t$ has a direct impact in the solution as it can be seen in *Figure 4*.

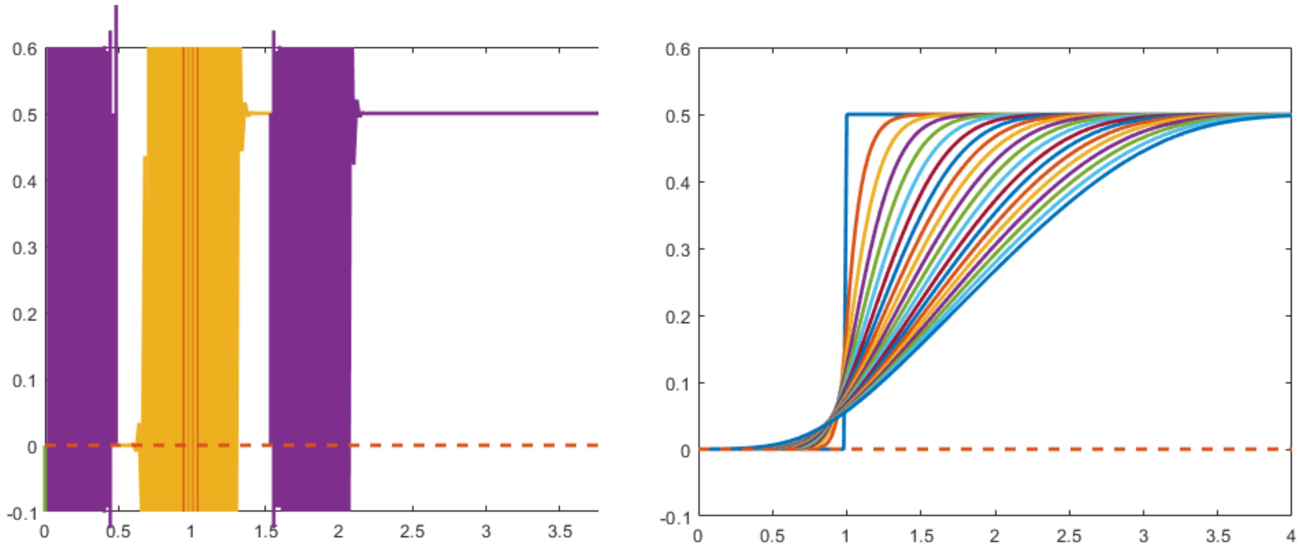


**Figure 4.** Numerical solution for $t_f=4s$, $\Delta t=0.1s$ and numerical diffusion $\varepsilon=10^{-2}$ for explicit method (left) and implicit methods (right).

The previous figure shows a fact that must be remarked. In the first case ($t_f$=4s, Δt=0.005s and numerical diffusion ε=$10^{-2}$) the explicit method worked fine because diffusion was high enough. However, now, even having the same diffusion, the solution 'blows up' for the explicit method given that the time step is not small enough.

Finally, for parameters $t_f$=4s, Δt=0.005s and ε=$10^{-4}$.

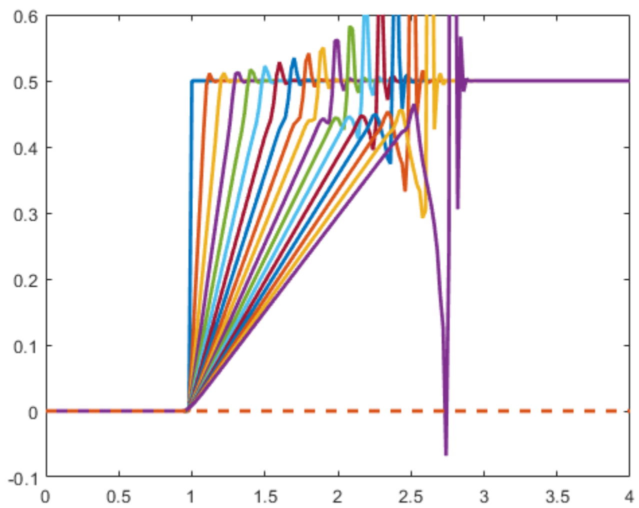


**Figure 5.** Numerical solution for explicit method for Δt=0.005s and ε=$10^{-4}$.

Lets consider now a new initial configuration as shown in *Figure 6.*

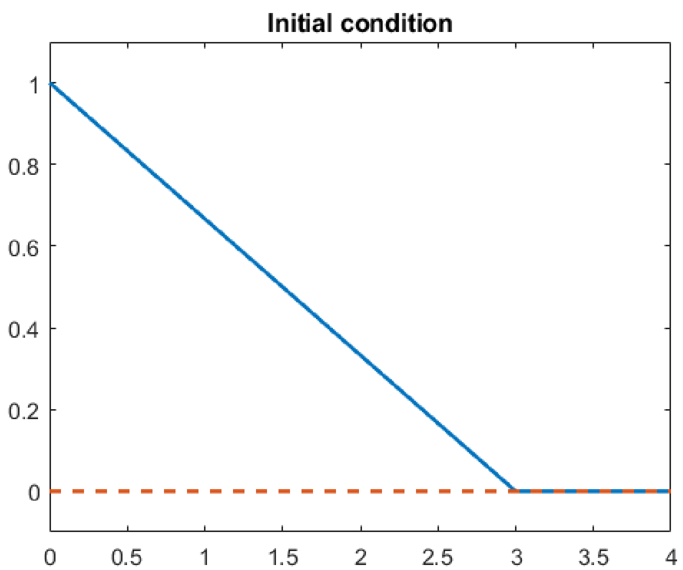


**Figure 6.** Initial configuration of our solution.

Now, for parameters $t_f$=4s, $\Delta t$=0.005s and numerical diffusion $\varepsilon=10^{-2}$

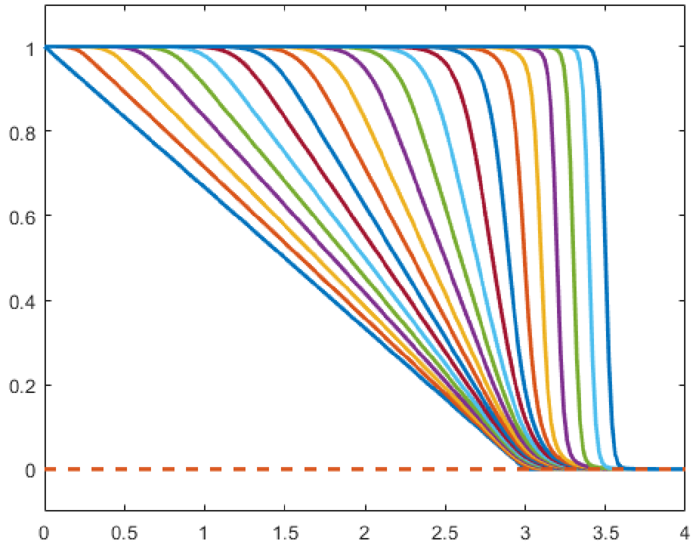


**Figure 7.** Evolution of the solution for Euler, implicit Picard and Newton-Rapson each 0.1 seconds.

From *Figure 7*. it can be seen how the solution evolves and a shock-wave appears. A shock wave is a wave where there in an abrupt change of a physical quantity. Again, with those parameters the solution is good for all methods. However, if we change parameters and choose $\varepsilon=10^{-3}$ the same behaviour as before is observed. For the explicit method the solution 'blows up' for implicit method the solution is still fine.

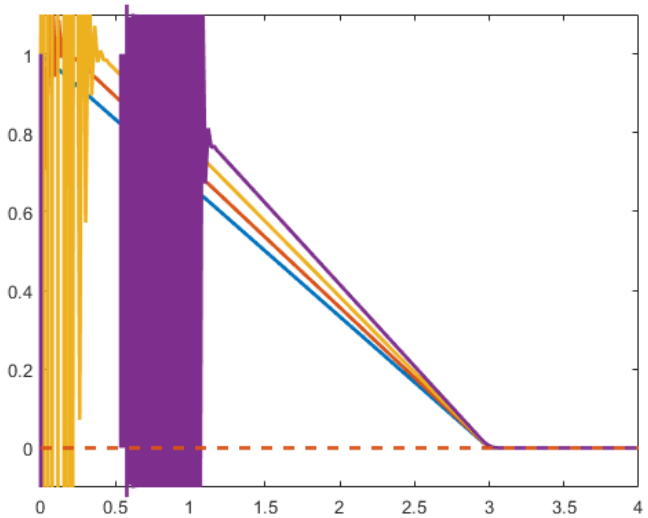


**Figure 8.** Numerical solution for $\Delta t$=0.05 and with numerical diffusion $\varepsilon=10^{-2}$ for explicit method.

To conclude, for the forth case where $t_f$=4s, $\Delta t$=0.005s and numerical diffusion $\varepsilon=10^{-2}$ implicit methods still worked, however, for this new problem the solution oscillates, as seen in *Figure 9.*

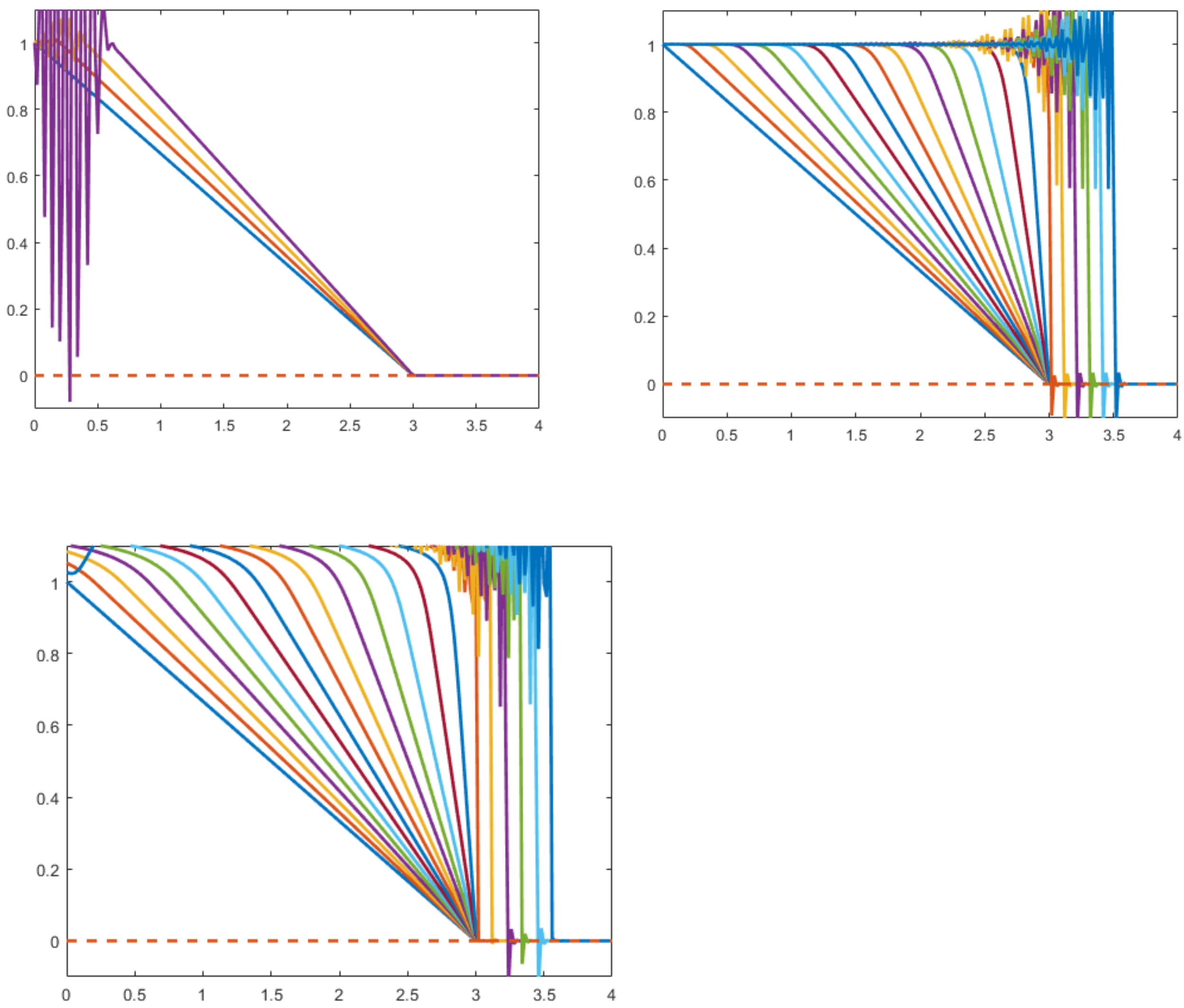


**Figure 9.** Numerical solution for explicit method (top left), implicit Picard (top right) and implicit Newton-Rapson (bottom).

For this problem, even implicit methods can't obtain a good solution although in the previous problem worked fine.

# Final Comments

In the previous work, burger's equation has been solved, although it was presented with the form:

$$u_t + u\,u_x = 0$$

this equation has no unique solution. What has been done in order to solve this equation was to introduce a diffusion:

$$u_t^\epsilon + u^\epsilon u_x^\epsilon = \epsilon\, u_{xx}^\epsilon$$

Finally, to end this report some comments must be done regarding the coding of this work. Picard method was already implemented but Newton-Rapson was still to be implemented. The only difference and what it has been changed is the way the solution at next time step is computed. Both methods compute it iteratively but the mathematical expression is different.

The iteration is computed for each method as follows:

$$^{k+1}\mathbf{U^{n+1}} = \mathbf{A}^{-1}(^{k}\mathbf{U^{n+1}})(\mathbf{M}\mathbf{U}^{n})$$

**(Picard method)**

where

$$\boxed{\mathbf{A}(\mathbf{U^{n+1}})} = \left(\mathbf{M} + \Delta t \left(\mathbf{C}(\mathbf{U}^{n+1}) + \epsilon \mathbf{K}\right)\right)$$

While Newton-Rapson:

$$^{k+1}\mathbf{U}^{n+1} = {}^{k}\mathbf{U}^{n+1} - \mathbf{J}^{-1}(^{k}\mathbf{U}^{n+1})\mathbf{f}(^{k}\mathbf{U}^{n+1})$$

where $\mathbf{J} = \dfrac{d\mathbf{f}}{d\mathbf{U}}$ is the Jacobian matrix

$$\mathbf{f}(\mathbf{U}) = (\mathbf{M} + \Delta t \mathbf{C}(\mathbf{U}) + \epsilon \Delta t \mathbf{K})\,\mathbf{U} - \mathbf{M}\mathbf{U}^{n}$$

**(Newton-Rapson method )**

Finally, it must be said that the algorithm which iterates the solution is different, however the solution should be the same if the error for which it stops iterating at each step is the same. The only difference will be the number of iterations until converging to the solution with desired error. Being Picard linear and Newton-Rapson quadratic.

This fact of equal solution has been seen through this report in the figures, where for all problems and parameters the solution was identical for both methods reason for which only one plot for both method has been presented.