

Ordinary PDE

$$\begin{cases} U_t^E + U^E U_x^E = \varepsilon U_{xx}^E \\ U(x,0) = U(x) \\ U(-1,t) = U(1,t) = 0 \end{cases}$$

let's consider a second-order expansion in time of the unknown variable U .

$$U^{E+1} = U^E + \Delta t U_t^E + \frac{1}{2} \Delta t^2 U_{tt}^E + O(\Delta t^3)$$

$$\left(\frac{U^{E+1} - U^E}{\Delta t} = U_t^E + \frac{1}{2} \Delta t U_{tt}^E \right) + O(\Delta t^3) \quad \text{neglected}$$

from the governing equation we know that:

$$U_t^E = \varepsilon U_{xx}^E - U^E U_x^E \equiv f(u)$$

then if we derive this expression:

$$U_{tt}^E = (\varepsilon U_{xx}^E - U^E U_x^E)_t = \frac{df}{du} \frac{du}{dt} \quad \text{with} \quad \frac{df}{du} = -U_x^E$$

$$U_{tt}^E = (\varepsilon U_{xx}^E - U^E U_x^E)_t = U_{xx}^E + \varepsilon U_{xxxt}^E - U_t^E U_x^E - U^E U_{xt}^E =$$

$$= U_{xx}^E + \varepsilon (U_t^E)_{xx} - U_t^E U_x^E - U^E (U_t^E)_x \quad \text{using} \quad U_t^E = \varepsilon U_{xx}^E - U^E U_x^E$$

$$\begin{aligned}
 U_{tt}^E &= U_{xx}^E + \varepsilon \left[\varepsilon U_{xx}^E - U^E U_x^E \right]_{xx} - \left(\varepsilon U_{xx}^E - U^E U_x^E \right) U_x^E - U^E \left[\varepsilon U_{xx}^E - U^E U_x^E \right]_x = \\
 &= U_{xx}^E + \varepsilon^2 U_{xxxx}^E - \left(\varepsilon U_x^E U_{xx}^E + \varepsilon U^E U_{xx}^E \right)_x - \varepsilon U_{xx}^E U_x^E + U^E U_x^E{}^2 - U^E \varepsilon U_{xxx}^E + U^E U_x^E{}^2 + U^E U_{xx}^E{}^2 \\
 &= U_{xx}^E + \varepsilon^2 U_{xxxx}^E - \varepsilon U_{xx}^E U_x^E - \varepsilon U_x^E U_{xx}^E + \varepsilon U_x^E U_{xx}^E + \varepsilon U_x^E U_{xxx}^E - \varepsilon U_{xx}^E U_x^E + U^E U_x^E{}^2 \\
 &\quad - U^E \varepsilon U_{xxx}^E + U^E U_x^E{}^2 + U^E U_{xx}^E{}^2
 \end{aligned}$$

$$\boxed{= U_{xx}^E + \varepsilon^2 U_{xxxx}^E - 4\varepsilon U_x^E U_{xx}^E - 2\varepsilon U^E U_{xxx}^E + 2U^E U_x^E{}^2 + U^E{}^2 U_{xx}^E = U_{tt}^E}$$

So now we have all the ingredients:

$$\boxed{\frac{U^{E+1} - U^E}{\Delta t} = \varepsilon U_{xx}^E - U^E U_x^E + \frac{1}{2} \Delta t \left[U_{xx}^E + \varepsilon^2 U_{xxxx}^E - 4\varepsilon U_x^E U_{xx}^E - 2\varepsilon U^E U_{xxx}^E + 2U^E U_x^E{}^2 + U^E{}^2 U_{xx}^E \right]}$$

2 final.