Universitat Politècnica de Catalunya

Master of Science in Computational Mechanics

Finite Elements in Fluids

HW2
Steady Transport

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1 Steady transport solution methods

The steady convective-diffusive transport without reaction and with Dirichlet boundary conditions is defined by the following equation:

$$R(u) = \boldsymbol{a} \cdot \nabla u - \nabla \cdot (\nu \nabla u) - s = 0 \qquad \text{in } \Omega \tag{1.1}$$

$$u = u_D \qquad \text{in } \Gamma_D \tag{1.2}$$

where u is the transported unknown property, $\boldsymbol{a}$ is the convection velocity, ν is the diffusivity coefficient, s is the source term, Ω is the domain and u_D represents the prescribed values on the boundary Γ_D.

1.1 Weak form

In order to discretize the domain we first attribute a weak form to the problem by multiplying the equation 1.2 by an arbitrary weighting function w which is zero on Γ_D. Yielding, after an integration over the domain:

$$\int_\Omega w(\boldsymbol{a} \cdot \nabla u) d\Omega - \int_\Omega w \nabla \cdot (\nu \nabla u) d\Omega = \int_\Omega w s d\Omega \tag{1.3}$$

Integrating the second term on the l.h.s by parts yields the weak form of the given problem:

$$\int_\Omega w(\boldsymbol{a} \cdot \nabla u) d\Omega + \int_\Omega \nabla w \cdot (\nu \nabla u) d\Omega = \int_\Omega w s d\Omega \tag{1.4}$$

1.2 Discretization

To solve equation 1.4 numerically, the unknown u is approximated by a piece-wise polynomial expression and the domain is discretized, as shown on equation 1.5

$$u \approx \sum_{i=1}^{n+1} N_i u_i \qquad x = \sum_{i=1}^{n+1} x_i N_i(\xi) \tag{1.5}$$

where n is the number of elements in the discretization, u_i is a constant value of the unknown at the node i (and, thus, not affected by the derivatives in equation 1.4) and N is the shape function that must have value one on node i and zero on all other nodes. The Galerkin formulation imposes also that the weighting function is given by the shape function $w = N_i$. The latter can have different forms, being the two simplest, lineal and quadratic. Considering a 1-D domain, they are given by equations 1.6 and 1.7, respectively.

$$N_1 = \frac{1}{2}(1-\xi) \qquad N_2 = \frac{1}{2}(1+\xi) \tag{1.6}$$

$$N_1 = \frac{1}{2}\xi(\xi-1) \qquad N_2 = 1-\xi^2 \qquad N_3 = \frac{1}{2}\xi(\xi+1) \tag{1.7}$$

Equations 1.6 and 1.7 are given in the isoparametric form for generalization (with $\xi \in [-1, 1]$), where an "reference element" with local coordinates is used and then mapped to the global coordinates by change of variables.

The previous approximation and discretization allows the description of the problem as the linear system presented in equation 1.8:

$$(C + K)u = f \tag{1.8}$$

where C, K and f are given by equations 1.9, 1.10 and 1.11.

$$C_{ij} = \int_\Omega N_i(\boldsymbol{a} \cdot \nabla N_j) d\Omega \tag{1.9}$$

$$K_{ij} = \int_\Omega \nabla N_i \cdot (\nu \nabla N_j) d\Omega \tag{1.10}$$

$$f_i = \int_\Omega N_i s d\Omega \tag{1.11}$$

On the linear case the shape functions provide the following change of variables:

$$x = \frac{x_2 - x_1}{2}\xi + \frac{x_1 + x_2}{2} \tag{1.12}$$

Thus, the jacobian for the mapping of the reference element with the properties $\nabla_x = J^{-1}\nabla_\xi$ and $dx = |J|d\xi$ is given by

$$J = \frac{\partial x}{\partial \xi} = \frac{x_2 - x_1}{2} \tag{1.13}$$

1.3 Stability

For a regular mesh with constant element size $x_j - x_i = h$ the Galerkin method stability is restricted to the Péclet number $Pe = |a|h/2\nu < 1$, meaning that the mesh refinement must strictly follow the relation between the convection velocity and the diffusion coefficient in order to maintain stability, sometimes imposing impractically refined grids to the solver. This limitation stimulated the creation of stabilization techniques such as the Streamline Upwind (SU), the Streamline Upwind Petrov-Galerkin (SUPG) and the Galerkin Least-Squares (GLS). The first introduces an artificial diffusion with a stabilization parameter $\tau = \bar{\nu}/||a||^2$ as following:

$$\int_\Omega w(\boldsymbol{a} \cdot \nabla u) d\Omega + \int_\Omega \nabla w \cdot (\nu \nabla u) d\Omega + \int_\Omega \tau (\boldsymbol{a} \cdot \nabla w)(\boldsymbol{a} \cdot \nabla u) d\Omega = \int_\Omega w s d\Omega \tag{1.14}$$

For a 1-D convection-diffusion problem, the optimal stabilization parameter is given by:

$$\bar{\nu} = \beta \frac{ah}{2} \qquad \beta = coth(Pe) - \frac{1}{Pe} \tag{1.15}$$

This method, though, is not consistent, leading to inaccuracy on nodal solutions when the source term is not constant. On the other hand, the other two methods, namely SUPG and GLS, are based on the original equation 1.2. Since $R(u)$ is intrinsically zero, these methods are consistent. They are given by:

$$\int_\Omega w(\boldsymbol{a} \cdot \nabla u) d\Omega + \int_\Omega \nabla w \cdot (\nu \nabla u) d\Omega + \sum_e \int_{\Omega_e} P(w) \tau R(u) d\Omega = \int_\Omega w s d\Omega \tag{1.16}$$

where for SUPG:

$$P(w) = \boldsymbol{a} \cdot \nabla w \tag{1.17}$$

and for GLS:

$$P(w) = \boldsymbol{a} \cdot \nabla w - \nabla \cdot (\nu \nabla w) \tag{1.18}$$

1.4 MATLAB implementation

Given the complete code for the Galerkin and SU formulation with linear interpolation, the SUPG and GLS algorithm were incremented according to the equations in the previous section. Four combinations of convection velocity, diffusion coefficient and mesh sizes were tested, as presented on Table 1.1

Table 1.1: Test cases

	$\boldsymbol{a}$	ν	N° of Elements
Case 1	1	0.2	10
Case 2	20	0.2	10
Case 3	1	0.01	10
Case 4	1	0.01	50

1.4.1 Linear approximation with no source term

Initially, the convective-diffusive transport was evaluated with the linear shape functions stated on equation 1.6 and with a source term $s = 0$. The Galerkin method was used to solve all cases yielding the results on Figure 1.1

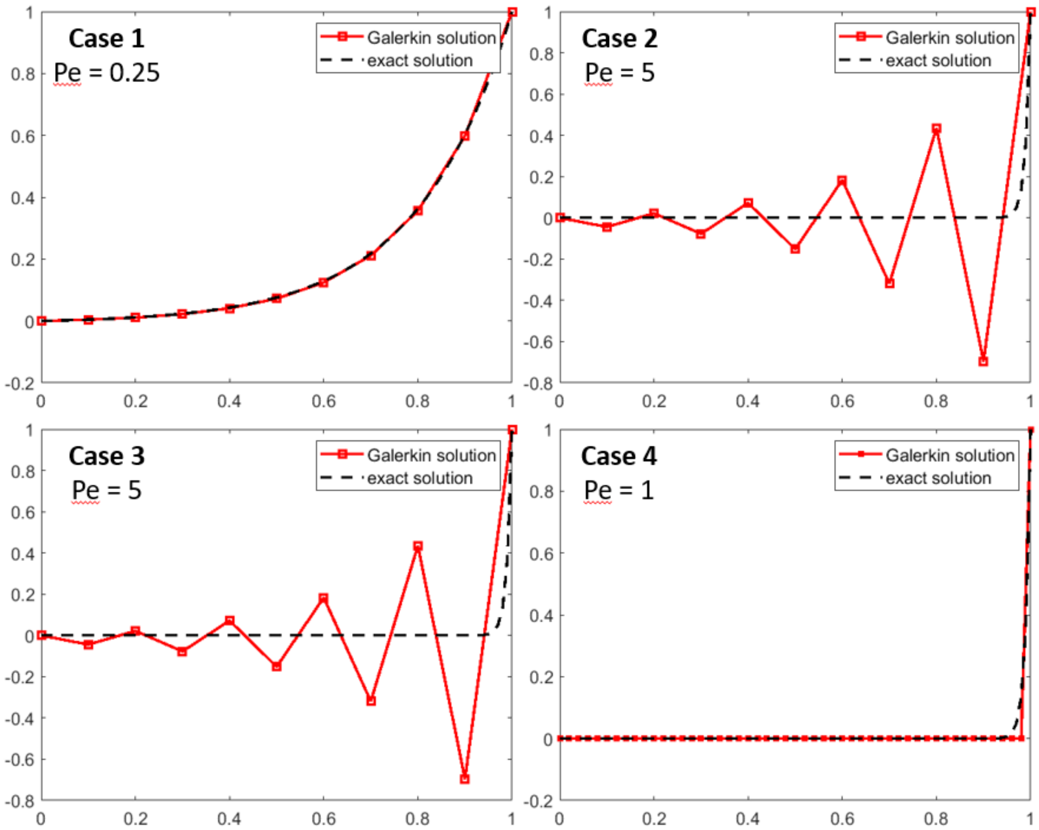

Figure 1.1: Galerkin solution with no source term and linear shape functions

As expected, the stability limitations of the pure Galerkin method stated on the previous section are evident on cases 2 and 3, in which the Péclet number was over 1. The effectiveness of the stabilization methods are presented on Figure 1.2, taking the case 3 as reference.

As seen, all methods were capable to stabilize the Galerkin method and reach the exact solution on nodal points.

1.4.2 Linear approximation with variable source term

To further investigate the capabilities of the methods, a variable source term $s = 10e^{-5x} - 4e^{-x}$ is introduced.

The behaviour of the Galerkin method is presented for all cases on Figure 1.3 and the comparisons between the other methods are shown on Figure 1.4.

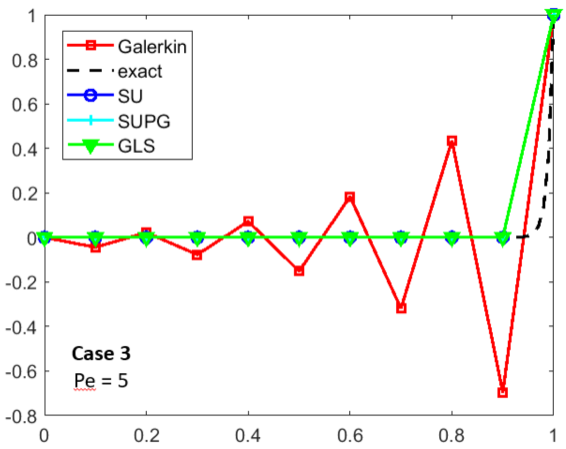

Figure 1.2: Methods comparison with no source term and linear shape functions

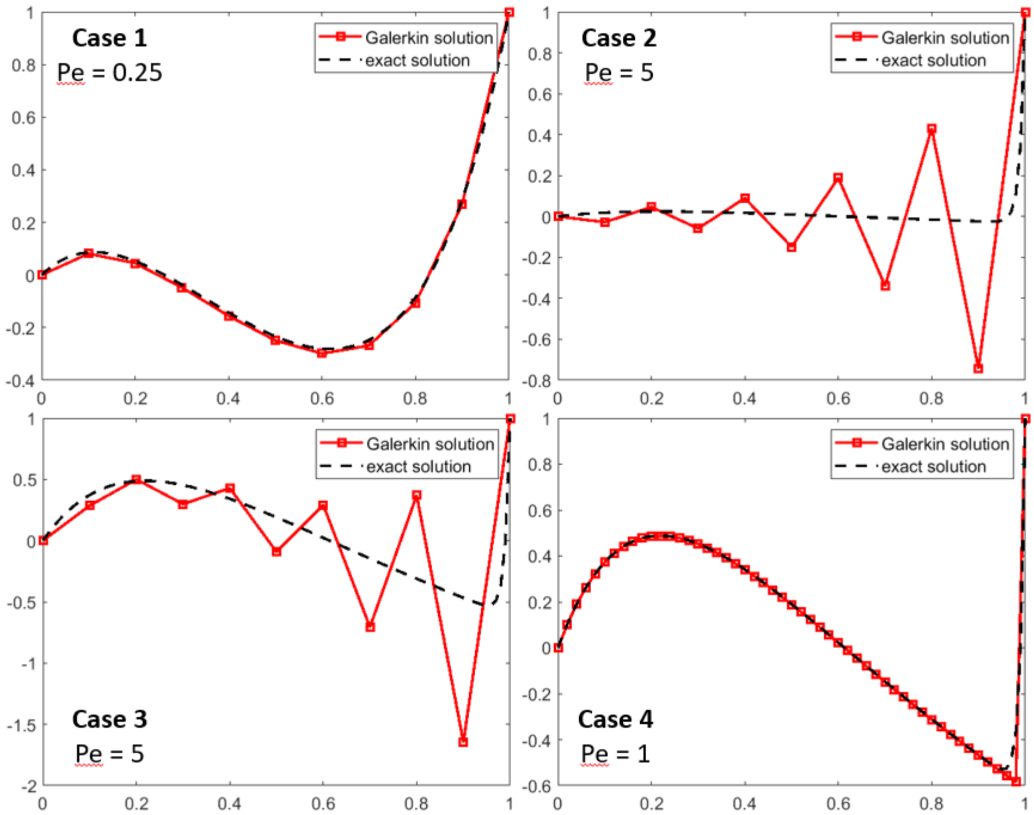

Figure 1.3: Galerkin solution with source term and linear shape functions

Again, the Galerkin method shows a growing oscillation for $Pe > 1$. However, even though all other methods were able to stabilize the problem, the SU solution is displaced when compared to the exact solution. This is due to its inconsistency (added diffusion is not related to the residual), being, thus, unable to reach nodal exact values under

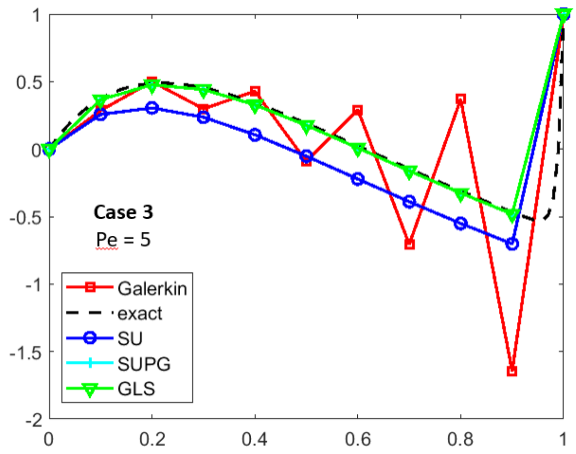

Figure 1.4: Methods comparison with source term and linear shape functions

variable source terms.

This evaluation points to the importance of the consistent formulation of SUPG and GLS. Although the SU result is smooth, stable and "apparently correct" it is considerably inaccurate. Provided that on most of the applications the exact solution is not known for later comparison, one should take into account the limitations and dangers of the SU formulation.

1.4.3 Quadratic approximation with variable source term

Finally, the quadratic shape functions were added to the initial code and tested and tested through the same parameters. The code implementation required a different treatment of the stabilization parameter, assigning a matrix to it in order to cover the different approaches for the nodes on the middle and on the edges of the element as described in [1].

The results for the Galerkin method is presented for all cases on Figure 1.5 and the comparisons between the other methods are shown on Figure 1.6.

It stands out that, although the Galerkin method still provides unstable results for $Pe > 1$, the oscillations are smaller when compared to the solutions presented on Figures 1.1 or 1.3. This happens because the quadratic method, in practice, introduces more nodes to the calculation using the same number of elements. Therefore, the actual Péclet number is in fact smaller (half of the one got from a linear approximation).

Once more the SU method provided a displaced solution, while SUPG and GLS behaved well for the case. Also worth noticing is the somewhat worst treatment of the boundary layer at the Dirichlet boundary condition at $x = 1$. The sudden change of the function couldn't be well represented by a parabolic function as well as a linear function with the used spatial discretization.

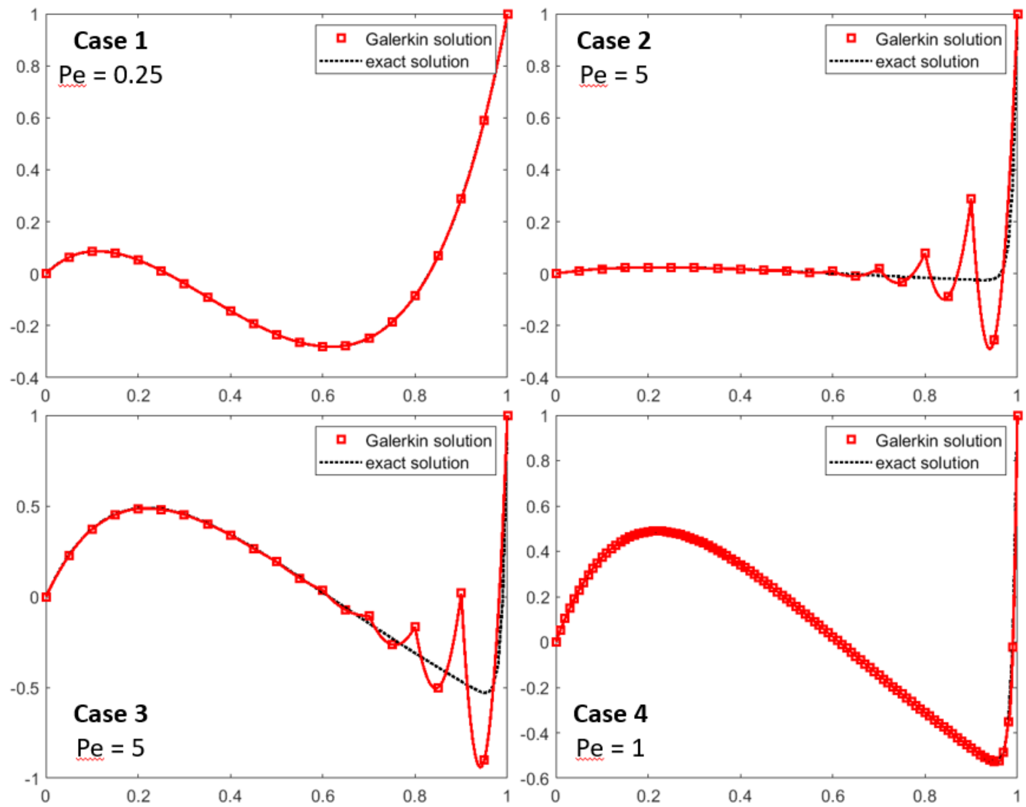

Figure 1.5: Galerkin solution with source term and parabolic shape functions

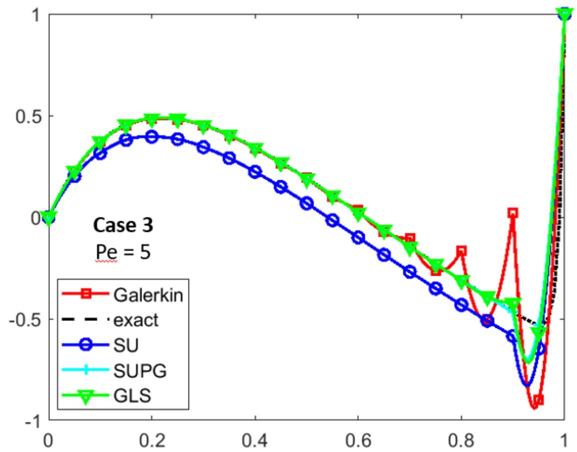

Figure 1.6: Methods comparison with source term and parabolic shape functions

Bibliography

[1] J. Donea and A. Huerta. *Finite Element Methods for Flow Problems*. Finite Element Methods for Flow Problems. John Wiley & Sons, 2003.