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Assignment 2

1 Problem statement

In this report, the author compared different schemes for solving steady transport problems – standard Galerkin, SU, SUPG and GLS. The strong form of the problem is written as

$$\begin{cases} \boldsymbol{a} \cdot \boldsymbol{\nabla} u - \boldsymbol{\nabla} \cdot (\nu \boldsymbol{\nabla} u) = s & \mathbf{x} \in \Omega \\ u(\mathbf{x}) = u_D & \mathbf{x} \in \Gamma_D \end{cases} \tag{1}$$

In the following sections, different examples are shown based on different definition of the source term and the boundary values.

2 Maths derivation

The weak form associated with this model problem is, after integration by parts of the diffusion term, given by

$$\int_\Omega w(\boldsymbol{a} \cdot \boldsymbol{\nabla} u) + \boldsymbol{\nabla} w \cdot (\nu \boldsymbol{\nabla} u) d\Omega = \int_\Omega w s d\Omega \tag{2}$$

The weak form will be discretised with shape functions. At last we obtain a linear system

$$\mathbf{Ku} = \mathbf{f} \tag{3}$$

where

$$\mathbf{K}_{ij} = \int_{\Omega_e} N_i(\boldsymbol{a} \cdot \boldsymbol{\nabla} N_j) d\Omega + \int_{\Omega_e} \boldsymbol{\nabla} N_i \cdot (\nu \boldsymbol{\nabla} N_j) \tag{4a}$$

$$\mathbf{f}_p = \int_{\Omega_e} N_i s d\Omega \tag{4b}$$

The general form of the stabilization techniques is

$$\int_\Omega w(\boldsymbol{a} \cdot \boldsymbol{\nabla} u) + \boldsymbol{\nabla} w \cdot (\nu \boldsymbol{\nabla} u) d\Omega + \sum_e \int_{\Omega_e} \mathcal{P}(w) \tau \mathcal{R}(u) d\Omega = \int_\Omega w s d\Omega \tag{5}$$

where $\mathcal{P}(w)$ is a certain operator applied to the test function and $\mathcal{R}$ is the residual of the differential equation

$$\mathcal{R}(u) = \boldsymbol{a} \cdot \boldsymbol{\nabla} u - \boldsymbol{\nabla} \cdot (\nu \boldsymbol{\nabla} u) - s = \mathcal{L}(u) - s \tag{6}$$

In the SUPG method, $P(w) = \boldsymbol{a} \cdot \boldsymbol{\nabla} w$. Follow the same discretization procedure

$$\mathbf{K}_{ij} = \int_{\Omega_e} N_i(\boldsymbol{a} \cdot \boldsymbol{\nabla} N_j) d\Omega + \int_{\Omega_e} \boldsymbol{\nabla} N_i \cdot (\nu \boldsymbol{\nabla} N_j) + \int_{\Omega_e} (\boldsymbol{a} \cdot \boldsymbol{\nabla} N_i) \tau (\boldsymbol{a} \cdot \boldsymbol{\nabla} N_j - \nu \nabla N_j) d\Omega_e \tag{7a}$$

$$\mathbf{f}_i = \int_{\Omega_e} N_i s d\Omega + \int_{\Omega_e} \boldsymbol{a} \cdot \boldsymbol{\nabla} N_i s d\Omega_e \tag{7b}$$

In the GLS method, $P(w) = \boldsymbol{a} \cdot \boldsymbol{\nabla} w - \boldsymbol{\nabla} \cdot (\nu \boldsymbol{\nabla} w)$, which introduce symmetry to the final discretized formulation

$$\begin{aligned}
\mathbf{K}_{ij} &= \int_{\Omega_e} N_i(\boldsymbol{a} \cdot \boldsymbol{\nabla} N_j) d\Omega + \int_{\Omega_e} \boldsymbol{\nabla} N_i \cdot (\nu \boldsymbol{\nabla} N_j) \\
&+ \int_{\Omega_e} (\boldsymbol{a} \cdot \boldsymbol{\nabla} N_i - \nu \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} N_j) \tau (\boldsymbol{a} \cdot \boldsymbol{\nabla} N_j - \nu \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} N_j) d\Omega_e \\
\mathbf{f}_i &= \int_{\Omega_e} N_i s d\Omega + \int_{\Omega_e} (\boldsymbol{a} \cdot \boldsymbol{\nabla} N_i - \nu \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} N_j) s d\Omega
\end{aligned}$$

The global system is obtained by assembling each element matrix.

3 Standard Galerkin method

3.1 Linear approximation

The first example is defined as $s = 0$ with boundary conditions: $u(0) = 0$ and $u(1) = 1$. This example is computed with linear elements using the Galerkin scheme for several values of the mesh Péchlet number.

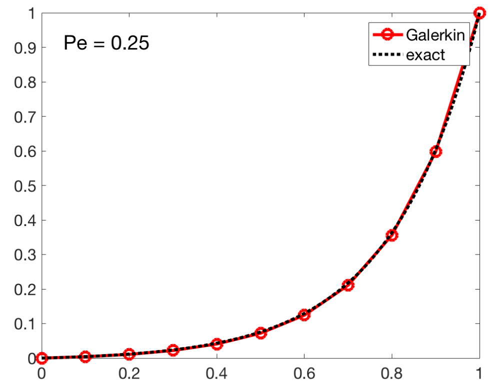
- (a) $a = 1$, $\nu = 0.2$, 10 elements
- (b) $a = 20$, $\nu = 0.2$, 10 elements
- (c) $a = 1$, $\nu = 0.01$, 10 elements
- (d) $a = 1$, $\nu = 0.01$, 50 elements

The results are displayed in Figure 1 in comparison with the exact solution. It shows that the Galerkin solution is corrupted by non-physical oscillations when the Péchlet number is larger than one. This problem can be solved by refining the mesh. However it is computational expensive and it has the truncation error.

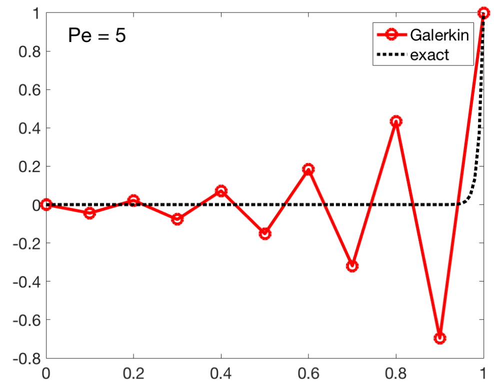
3.2 Quadratic approximation

The implementation of quadratic approximation includes the following key points:
- The shape functions
- The elemental length is defined as the distance of every two nodes
- Different stabilization parameters are used for the corner nodes and the mid-side nodes
- The values at the inner nodes need to be interpolated

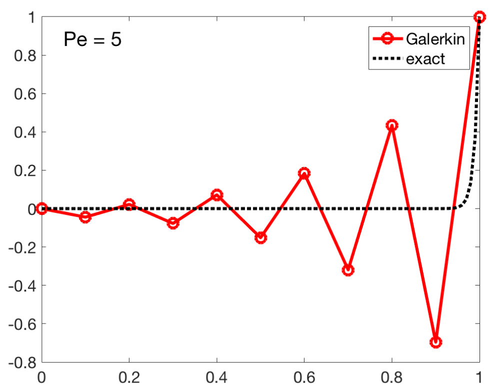
Another test for quadratic elements is performed by solving case (a) and case (c) in Section 3.1. The results are shown in Figure 2. Again the Galerkin solution is characterised by spurious node-to-node oscillations when convective effects becomes important. In conclusion, the Galerkin method is not optimal methods for solving convection-dominated problems. Therefore we shall look for some techniques to remedy the lack of stability of the Galerkin finite element method and thereby obtain stable and accurate approximation to the convection-dominated problems.



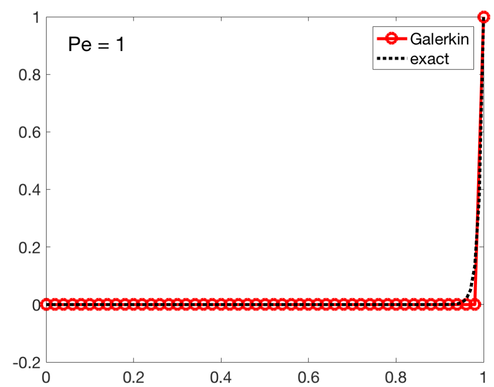
(a) Pe=0.25



(b) Pe=5

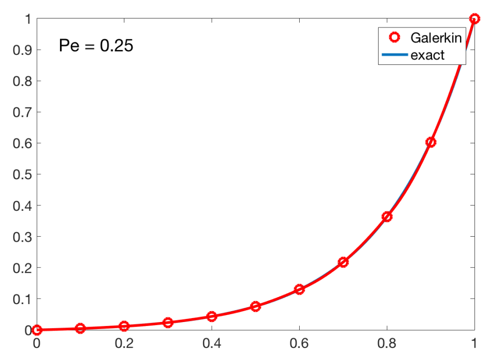


(c) Pe=5

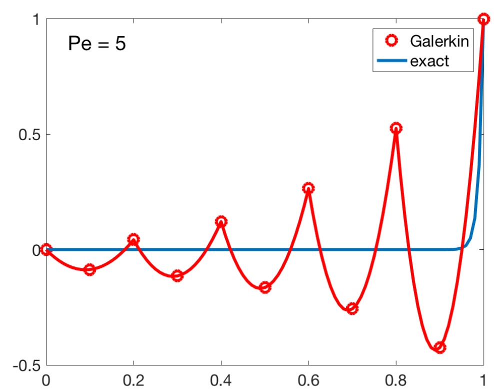


(d) Pe=1

Figure 1: Problem 1 solved with linear standard Galerkin

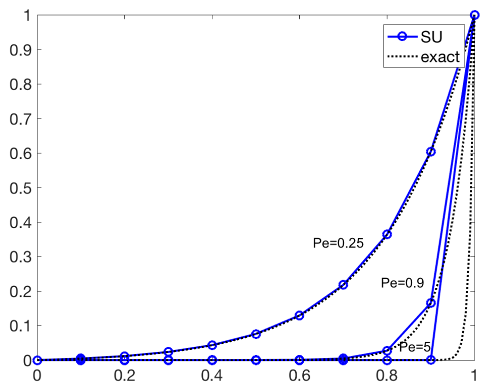


(a) Pe=0.25

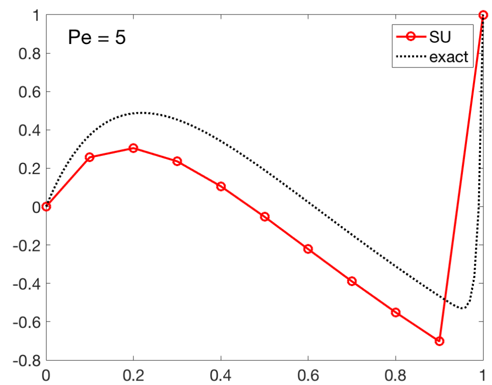


(b) Pe=5

Figure 2: Quadratic standard Galerkin



(a) constant source



(b) nonconstant source

Figure 3: Streamline Upwind

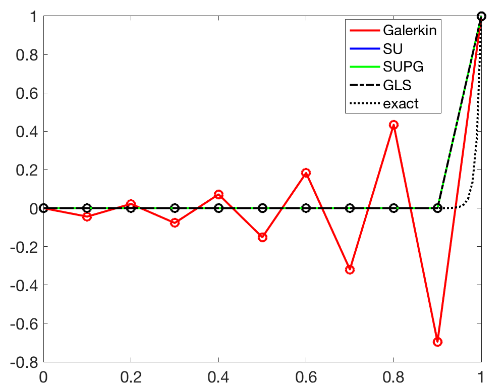
4 Stabilization techniques

4.1 Streamline Upwind

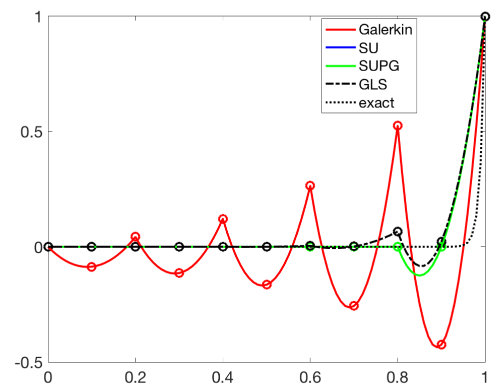
The first example above are solved with the Streamline Upwind method under different values of Péchlet number. Figure 3 illustrates the results of constant source problem and nonconstant source one computed by Streamline Upwind. The nonconstant source term is defined as $s = 10 * exp(-5 * x) - 4 * exp(-x)$ with the same boundary conditions. The optimal approximation producing the exact solution at the nodes of a uniform mesh of linear elements may be obtained using the recommended coefficient for the stabilization. However, if the source term is not constant, the solution at the nodes are not exact.

4.2 SUPG and GLS

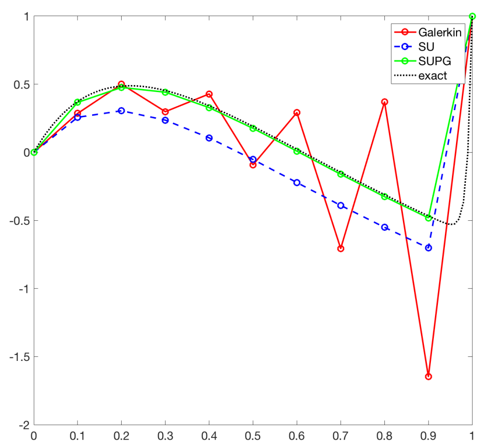
In order to produce more accurate solution at the nodes, we use different stabilization parameters for the corner nodes and the mid-side nodes. Different numerical methods are compared with the first example case (c) and another convection diffusion problem with $s = 10 * exp(-5 * x) - 4 * exp(-x)$ and $u(0) = 0$, $u(1) = 1$ as boundary conditions. Figure 4 demonstrates the solutions with different numerical methods using linear and quadratic meshes. It can be observed that SU, SUPG and GLS can reproduce exact solution with the optimal stabilization parameters in Ex1 case (c) with linear meshes. However it is not true for quadratic meshes and nonconstant source cases. There is less vibration in SUPG campared with the other methods. In conclusion, the SUPG formulation performs better than the original SU technique because it is consistent . There is no major difference between SUPG and GLS methods from a practicle point of view. In fact, both methods are identical for linear element problems.



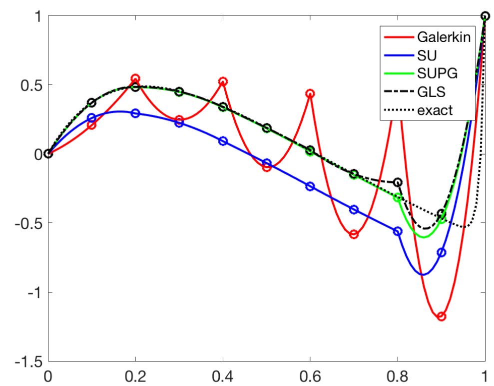
(a) First example case (c) Linear



(b) First example case (c) Quadratic



(c) Unconstant source Linear



(d) Unconstant source Quadratic

Figure 4: SU, SUPG and GLS solutions of the convection diffusion proglem at $P_e = 5$ using a uniform mesh of linear elements (left) and five quadratic elements (right)