Steady convection - diffusion equation

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Introduction

The convection-diffusion equation is the equation which governs the physical phenomena of transport of a pollutant in the air, transport of ink in a fluid...

It's equation reads:

$$\frac{\partial u}{\partial t} + \vec{a}\cdot\nabla u - \nabla\cdot(\nu\nabla u) = s$$

where: $\begin{cases} \vec{a}\cdot\nabla u \text{ is the convection term} \\ \nabla\cdot(\nu\nabla u) \text{ is the diffusion term} \\ s \text{ is the source term} \end{cases}$ and $\begin{cases} \vec{a} = \text{transport velocity} \\ \nu = \text{diffusivity} \end{cases}$

given that this problem is really difficult to solve analytically numerical methods must be used. In that case we will use finite elements and also the steady problem will be solved, it is $\frac{\partial u}{\partial t} = 0$

So our equation becomes:

$$\boxed{\vec{a}\cdot\nabla u - \nabla\cdot(\nu\nabla u) = s}$$

Weak form of steady convection-diffusion equation

To solve the problem numerically first of all the problem must be changed from strong to weak form:

$$\left.\begin{aligned} &\vec{a}\cdot\nabla u - \nabla\cdot(\nu\nabla u) = s && \text{in } \Omega \\ &u = u_D && \text{on } \Gamma_D \\ &\nu\frac{\partial u}{\partial n} = u_N && \text{on } \Gamma_N \end{aligned}\right\} \text{strong form.}$$

to move to weak form we multiply the differential equation by a test function w and integrate over the domain Ω

$$\int_\Omega w(\vec{a}\cdot\nabla u)\,d\Omega - \int_\Omega w\nabla\cdot(\nu\nabla u)\,d\Omega = \int_\Omega w s\,d\Omega \qquad \forall w$$

using the formula to integrate by parts:

$$\int_\Omega u\,\nabla\cdot\vec{v}\,d\Omega = -\int_\Omega \nabla u\cdot\vec{v}\,d\Omega + \int_{\partial\Omega} u\vec{v}\cdot\hat{n}\,d\Gamma$$

we can integrate the second term using the previous formula:

$$-\int_\Omega w\,\nabla\cdot(\nu\nabla u)\,d\Omega = \int_\Omega \nabla w\cdot(\nu\nabla u)\,d\Omega - \int_{\partial\Omega} w\,(\nu\nabla u)\cdot\hat{n}\,d\Gamma$$

Knowing that $(\nu\nabla u)\cdot\hat{n} = \nu\nabla_n u = u_N$ we can write the last term as:

$-\int_{\partial\Omega} w\,u_N\,d\Gamma$ and given that we impose the solution to be exact in the dirichlet boundary:

$$-\int_{\partial\Omega} w\, u_N \, d\Gamma \longrightarrow -\int_{\partial\Gamma_N} w\, u_N \, d\Gamma_N$$

So at the end we have:

$$\boxed{\int_\Omega w(\vec{a}\cdot\nabla u)\, d\Omega + \int_\Omega \nabla w \cdot (\nu \nabla u)\, d\Omega = \int_\Omega w s\, d\Omega + \int_{\Gamma_N} w\, u_N \, d\Gamma}$$

which is the weak form

In the case on 1-D with linear finite elements with constant lenght h the galerkin approximation reduces to:

$$a\,\frac{u_{i+1} - u_{i-1}}{2h} - \nu\,\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0$$

Defining Péclet number: $P_e = \frac{|a|h}{2\nu}$ we can write

$$(P_e - 1)\,u_{i+1} + 2u_i - (P_e + 1)\,u_{i-1} = 0$$

rearrenging terms:

$$\boxed{(u_{i+1} - u_i) = \frac{P_e + 1}{1 - P_e}\,(u_i - u_{i-1})}$$

Notice that when $P_e > 1$ $\frac{P_e + 1}{1 - P_e} < 0$ then oscillations appear in our solution as will be seen in the results.

Results

Using Galerkin's method the following problems have been solved:

[a=1, ν=0.2, 10 linear elements
a=20, ν=0.2, 10 linear elements
a=1, ν=0.01, 10 linear elements
a=1, ν=0.01, 50 linear elements]

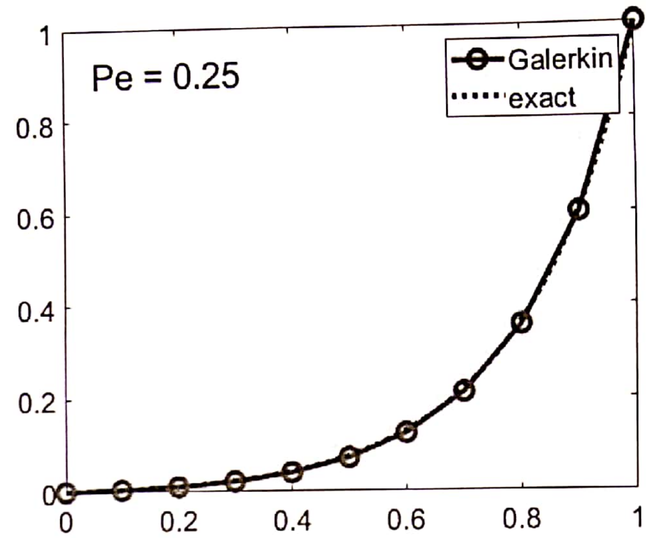


Figure 1. Numerical solution with Galerkin's method when a=1, ν=0.2 and 10 linear elements.

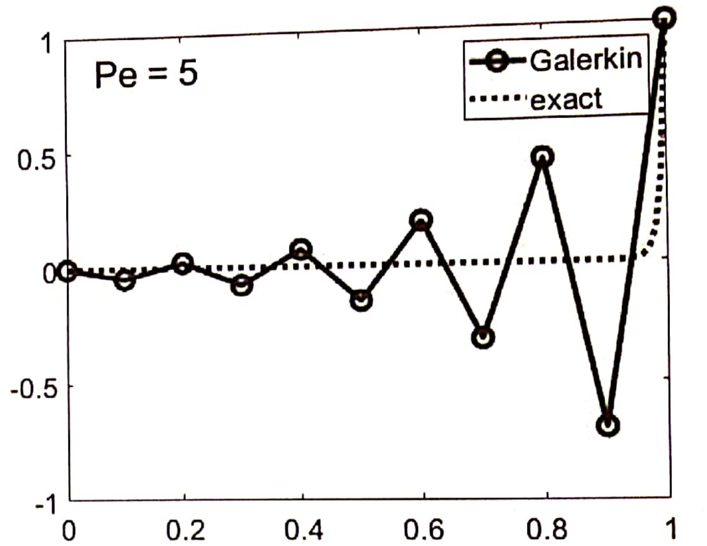


Figure 2. Numerical solution with Galerkin's method when a=20, ν=0.2 and 10 linear elements.

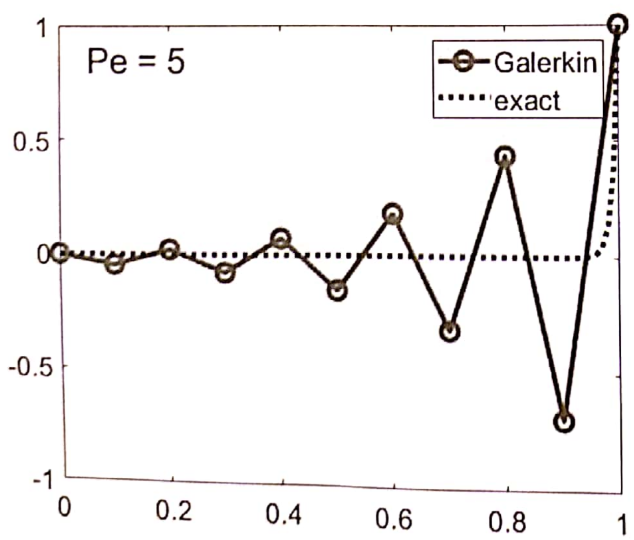


Figure 3. Numerical solution with Galerkin's method when a=1, ν=0.01 and 10 linear elements.

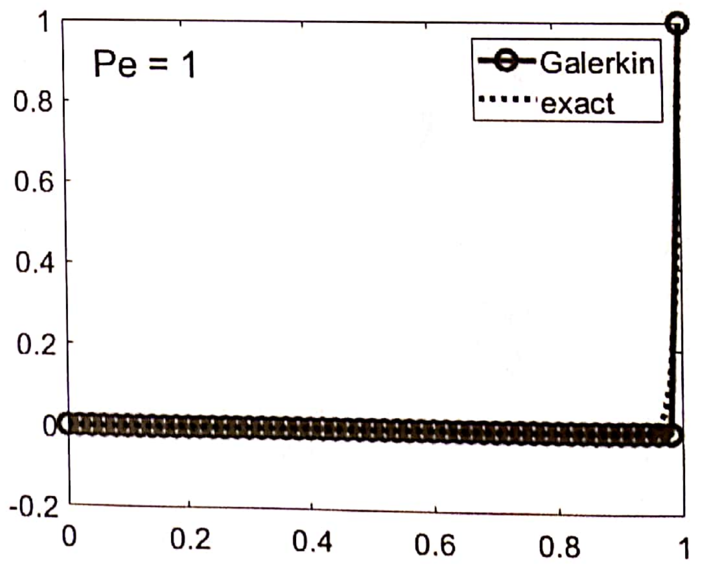


Figure 4. Numerical solution with Galerkin's method when a=1, ν=0.01 and 50 linear elements.

As it can be seen in the previous figures, Galerkin's method is not able to solve the problem when Péclet number (P_e) is greater than 1 $P_e > 1$ because the solution oscillates. It can also be seen that no matter if a and ν are different for two different problems as far as P_e has the same value, the solution will be the same (comparation between figure 2. and figure 3.)

In order to overcome this difficulty different methods are presented. Those methods deal with that problem "adding" more diffusion when convection dominates and adding extra terms compared with galerkin method to stabilize the solution.

The following methods have been used:

<u>Galerkin:</u>

The solution depends on:

$$K_{element} = \int_{\Omega^{(e)}} N \, a \, \frac{dN}{dx} \, d\Omega + \int_{\Omega^{(e)}} \frac{dN}{dx} \, \nu \, \frac{dN}{dx} \, d\Omega$$

where $N(x)$ are the shape functions (defined locally)

<u>SU:</u>

$$K_e = K_{galerkin} + \int_{\Omega^{(e)}} \tau \, a \, \frac{dN}{dx} \, a \, \frac{dN}{dx} \, d\Omega$$

$K_e = K_g$

<u>SUPG:</u>

$$K_e = K_{galerkin} + K_{SU} + -\int_{\Omega^{(e)}} \tau \, a \, \frac{dN}{dx} \, \nu \, \frac{d^2N}{dx^2} - \int_{\Omega^{(e)}} \tau \, a \, \frac{dN}{dx} \, s \, dx$$

GLS

$$K_{(e)} = K_{SUPG} + \int_{\Omega^e} \tau \cancel{\nu} \frac{d^2N}{dx^2}\left(a\frac{dN}{dx} - \nu\frac{d^2N}{dx^2} - s\right)$$

Solving our problem for case 3 (a=1, ν=0,01, 10 linear elements) the following results are obtained.

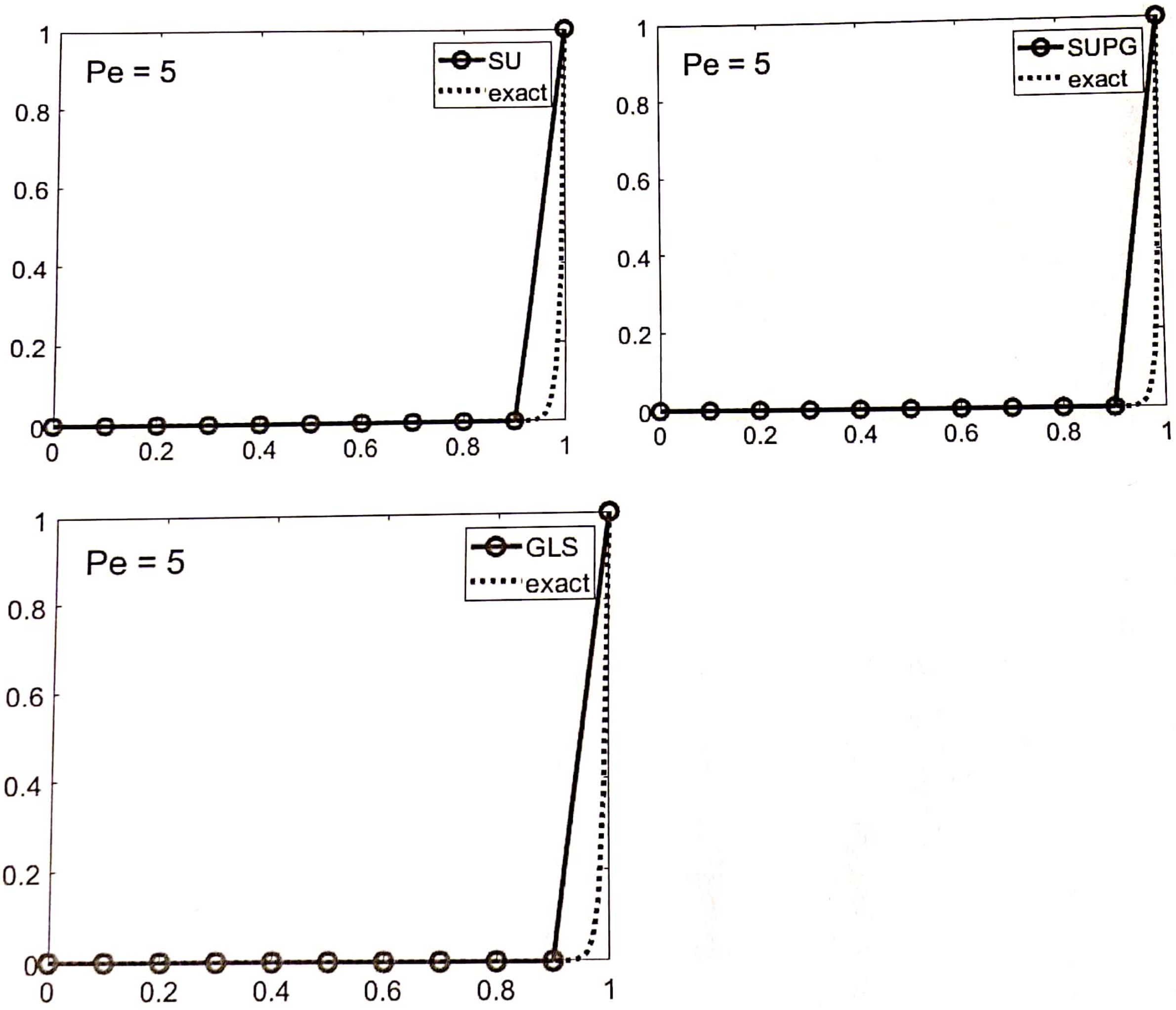


Figure 5. Numerical solution using SU (top left), SUPG (top right) and GLS (bottom) when a=1, ν=0.01 and 10 linear elements.

As it can be seen in Figure 5. all methods yield to the same solutions. The reason is that when considering linear elements the shape functions are:

$$N_1^{(e)}(x) = \frac{x_2 - x}{x_2 - x_1} \quad \text{and} \quad N_2^{(e)}(x) = \frac{x - x_1}{x_2 - x_1}$$

The different terms that make each method different from others vanishes because they imply second derivatives.

For example:

SU:

$$K_e = K_{galerking} + \int_{\Omega^{(e)}} \tau a \frac{dN_x}{dx} a \frac{dN_x}{dx} dx$$

SUPG:

$$K_e = K_{SU} + \int_{\Omega^{(e)}} \tau a \frac{dN}{dx} \left(a \frac{dN}{dx} - \nu \frac{d^2N}{dx^2} - S \right)$$

this term vanishes when considering linear elements, therefore $a u_x - S = R = 0$

So

$$K_{SUPG} = K_{SU} \text{ for linear elements.}$$

GLS:

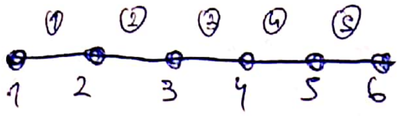
$$K_{GLS} = K_{SUPG} + \int \tau \nu \frac{d^2N}{dx^2} \left(a \frac{dN}{dx} - \nu \frac{d^2N}{dx^2} - S \right)$$

this term is zero

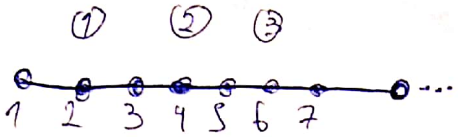
then $\boxed{K_{GLS} = K_{SUPG} = K_{SU}}$

To obtain different results quadratic elements must be cused.
To do so the code must be changed

First of all it must be noticed that the connectivity matrix is different for linear and quadratic elements.



linear elements $T = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ \vdots & \end{pmatrix}$



quadratic elements $T = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \\ & \vdots & \end{pmatrix}$

The number of nodes also changes by the following relation:

~~Num Eleme~~

$$\text{Num Nodes} = \text{Num Elements} \cdot p + 1 \qquad \text{being} \begin{cases} p=1 & \text{linear elements} \\ p=2 & \text{quadratic elements} \end{cases}$$

Finaly the position of the points also changes.

To finish our work the same problem as in case 3 is solved but with different source term.
In this case:

$$\boxed{f = 10\, e^{-5x} - 4\, e^{-x}}$$

$a=1$, $\nu = 0{,}01$ and 10 elements

To compare the numerical solution with the exact solution it must be computed. To obtain the exact solution the variation of constant methods is used given we know the analytical solution for the homogeneous problem

The results obtained for linear elements are:

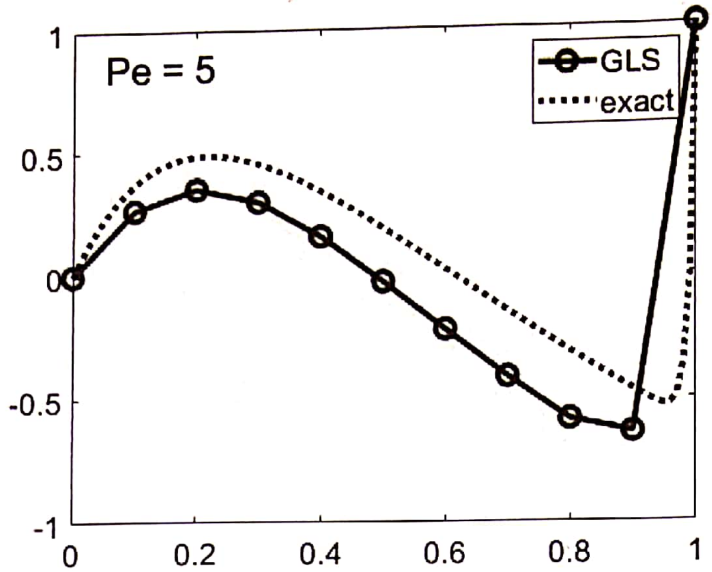


Figure 6. Numerical solution for the problem with the new source term when a=1, ν=0.01 and 10 linear elements.

For linear elements only the solution with GLS method is presented because other methods yields to the same solution.

Finally, in figure 7, 8, 9 the results with quadratic elements are presented. Here, the solution is different given that those different terms doesn't vanishes.

SU method is the one which performs worst. SUPG and GLS perform better, However they doesn't provide the exact solution.

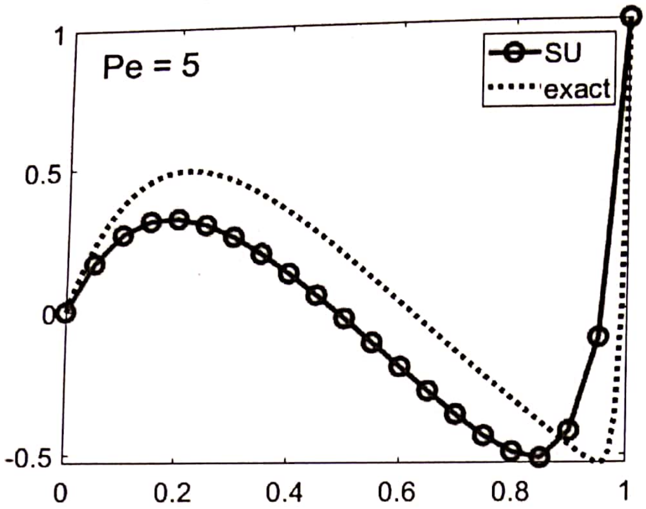


Figure 7. Numerical solution of the problem using SU method with quadratic elements.

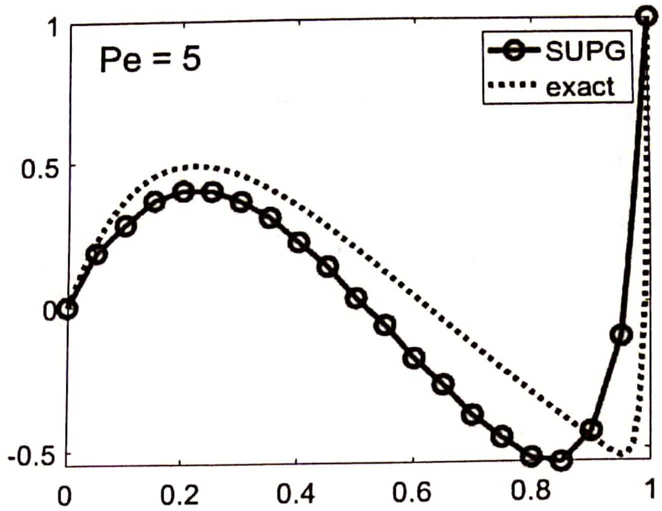


Figure 8. Numerical solution of the problem using SUPG method with quadratic elements.

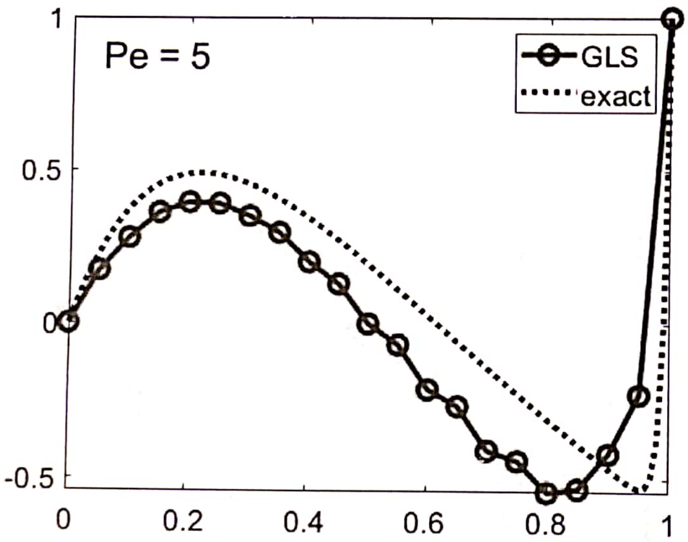


Figure 9. Numerical solution of the problem using GLS method with quadratic elements.