ELEFINFLUI: Finite Elements in Fluids	**(Due: 02/20/19)**

Assignment #1

Instructor: Antonio Huerta/Pablo Saez — *Name:* Mario Mendez Soto, *Id:* 03920395

Prove the identity of the conservative and non-conservative forms of the momentum conservation equation

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla\cdot\sigma = \rho\mathbf{b} \Leftrightarrow \frac{\partial(\rho\mathbf{v})}{\partial t} + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v} - \sigma) = \rho\mathbf{b}$$

Starting from the conservative form of the linear momentum equation, it is possible to derive the following:

$$\frac{\partial\rho\mathbf{v}}{\partial t} + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v} - \sigma) = \rho\mathbf{b}$$
$$\rho\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) - \nabla\cdot\sigma = \rho\mathbf{b}$$

The third term can be further simplified using the following matricial expression:

$$\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \rho uu & \rho uv & \rho uz \\ \rho vu & \rho vv & \rho vz \\ \rho wu & \rho wv & \rho ww \end{bmatrix}$$

The result of the operation will be a vector, whose x-component can be written as:

$$\frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} + \frac{\partial(\rho wu)}{\partial z}$$
$$\rho\frac{\partial(uu)}{\partial x} + uu\frac{\partial\rho}{\partial x} + \rho\frac{\partial(vu)}{\partial y} + vu\frac{\partial\rho}{\partial y} + \rho\frac{\partial(wu)}{\partial z} + wu\frac{\partial\rho}{\partial z}$$
$$\rho\Big(\frac{\partial(uu)}{\partial x} + \frac{\partial(vu)}{\partial y} + \frac{\partial(wu)}{\partial z}\Big) + u\Big(u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial x}\Big)$$
$$\rho\Big(u\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y} + u\frac{\partial w}{\partial z} + w\frac{\partial u}{\partial z}\Big) + u\begin{bmatrix} u & v & w \end{bmatrix}\cdot\begin{bmatrix} \frac{\partial\rho}{\partial x} & \frac{\partial\rho}{\partial y} & \frac{\partial\rho}{\partial z} \end{bmatrix}$$
$$\rho\Big(u\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\Big) + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\Big) + u(\mathbf{v}\cdot\nabla\rho)$$
$$\rho\Big(u\nabla\cdot\mathbf{v} + \begin{bmatrix} u & v & w \end{bmatrix}\cdot\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{bmatrix}\Big) + u(\mathbf{v}\cdot\nabla\rho)$$
$$\rho\Big(u(\nabla\cdot\mathbf{v}) + \mathbf{v}\cdot\nabla u\Big) + u(\mathbf{v}\cdot\nabla\rho)$$

Analogous expressions can be found for the y-component and z-component. As a result, the following identity is satisfied:

$$\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v}) = \rho\Big(\mathbf{v}(\nabla\cdot\mathbf{v}) + \mathbf{v}\cdot\nabla\mathbf{v}\Big) + \mathbf{v}(\mathbf{v}\cdot\nabla\rho)$$

Using the chain rule on the right-hand side of the equation, it yields:

$$\rho\Big(\mathbf{v}(\nabla\cdot\mathbf{v}) + \mathbf{v}\cdot\nabla\mathbf{v}\Big) + \mathbf{v}\Big(\nabla\cdot(\rho\mathbf{v}) - \rho(\nabla\cdot\mathbf{v})\Big)$$
$$\cancel{\rho\mathbf{v}(\nabla\cdot\mathbf{v})} + \rho\mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{v}\nabla\cdot(\rho\mathbf{v}) - \cancel{\rho\mathbf{v}(\nabla\cdot\mathbf{v})}$$
$$\rho\mathbf{v}\cdot\nabla\mathbf{v} + \mathbf{v}\nabla\cdot(\rho\mathbf{v})$$

Substituting into the original equation:

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v} \nabla \cdot (\rho \mathbf{v}) - \nabla \cdot \sigma = \rho \mathbf{b}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla)\mathbf{v} + \mathbf{v} \overbrace{\left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right)}^{\text{mass continuity}} - \nabla \cdot \sigma = \rho \mathbf{b}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla \cdot \sigma = \rho \mathbf{b}$$