FEF Assignment 1

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1 Problem Statement

The problem to be considered is a second-order linear scalar partial differential equation with given Neumann, Dirichlet and Robin boundary conditions along the boundary.

$$\begin{aligned} -\nabla.(\kappa\nabla u) &= s && in\ \Omega \\ u &= u_D && on\ \Gamma_D \\ n.(\kappa\nabla u) &= t && on\ \Gamma_N \\ n.(\kappa\nabla u) + \gamma u &= g && on\ \Gamma_R \end{aligned}$$

First, Ω is subdivided in n_{el} subdomains as follows:

$$\Omega = \bigcup_{i=1}^{n_{el}} \Omega_i, \quad \Omega_i \cap \Omega_j = \phi \quad for \quad i \neq j,$$

with boundaries $\partial\Omega_i$ defining an internal interface Γ

$$\Gamma = [\ \bigcup_{i=1}^{n_{el}} \partial\Omega_i\ ]\ \setminus\ \partial\Omega$$

The strong form can then be written in the broken domain as follows

$$\begin{aligned} -\nabla.(\kappa\nabla u) &= s && in\ \Omega_i \\ u &= u_D && on\ \Gamma_D \\ n.(\kappa\nabla u) &= t && on\ \Gamma_N \\ n.(\kappa\nabla u) + \gamma u &= g && on\ \Gamma_R \\ [\![un]\!] &= 0 && on\ \Gamma \\ [\![n.(\kappa\nabla u)]\!] &= 0 && on\ \Gamma \end{aligned}$$

where the last two equations deal with the continuity of u and its derivative along the internal boundary. The mixed form is then introduced.

$$\begin{aligned} \nabla\ .\ q &= s && in\ \Omega_i \\ q + (\kappa\nabla u) &= 0 && in\ \Omega_i \\ u &= u_D && on\ \Gamma_D \\ n.q &= -t && on\ \Gamma_N \\ n.q - \gamma u &= -g && on\ \Gamma_R \\ [\![un]\!] &= 0 && on\ \Gamma \\ [\![n.q]\!] &= 0 && on\ \Gamma \end{aligned}$$

This can be written in 2 problems: a local one given as:

$$\begin{aligned}
\nabla\ .\ q = s \qquad & in\ \Omega_i \\
q + (\kappa \nabla u) = 0 \qquad & in\ \Omega_i \\
u = u_D \qquad & on\ \Gamma_D \\
n.q = -t \qquad & on\ \Gamma_N \\
n.(q) - \gamma u = -g \qquad & on\ \Gamma_R \\
u = \hat{u} \qquad & on\ \partial\Omega_i\ \setminus\ \Gamma_D \cup \Gamma_N \cup \Gamma_R
\end{aligned}$$

The global one can then be written

$$[\![n.q]\!] = 0 \qquad on\ \Gamma$$

The numerical traces are then defined.

$$n.\hat{q}_i = \ \{ \quad \begin{array}{cc} n_i.q_i + \tau_i(u_i - u_D) & \partial\Omega \cap \partial\Gamma_D \\ n_i.q_i + \tau_i(u_i - \hat{u}) & \partial\Omega \cap \partial\Gamma \\ -t & \partial\Omega \cap \partial\Gamma_N \\ -g + \gamma u_i & \partial\Omega \cap \partial\Gamma_R \end{array}$$

The weak form of the local problem is obtained by multiplying by a weight function and integrating over the domain of n element. Integration by parts is done and the boundary conditions are added reaching the following form for the first equation.

$$\int_{\Gamma_R} v\gamma u d\Gamma + \int_{\Gamma\setminus\ \Gamma_N \cup \Gamma_R} v\tau_i u d\Gamma + \int_{\Omega_i} v\nabla.q d\Omega_i - \int_{\Gamma\cap(\Gamma_N\cup\Gamma_R)} vn.q d\Gamma =$$
$$\int_{\Omega_i} vs d\Omega_i + \int_{\Gamma_N} vt d\Gamma + \int_{\Gamma_R} vg d\Gamma + \int_{\Gamma_D} v\tau_i u_D d\Gamma + \int_{\Gamma\setminus\ \Gamma_N\cup\Gamma_R\cup\Gamma_D} v\tau_i \hat{u} d\Gamma$$

and the second equation

$$\int_{\Omega_i} \nabla.w\kappa u d\Omega_i - \int_{\Gamma_N} \kappa w.n\Gamma + \int_{\Gamma_R} \frac{\kappa}{\gamma}(w.n)(n.\kappa\nabla u)d\Gamma - \int_{\Omega_i} w.q\Omega_i =$$
$$\int_{\Gamma_D} \kappa u_D w.nd\Gamma + \int_{\Gamma_\Gamma\setminus\ \Gamma_N\cup\Gamma_R\cup\Gamma_D} \kappa\hat{u}w.nd\Gamma + \int_{\Gamma_R} \kappa\frac{g}{\gamma}w.nd\Gamma$$

For the global problem, the weak form is obtained as follows

$$\sum_{i=1}^{n_{el}} (\mu\tau_i u_i + \mu n.q - \mu\tau_i\hat{u})_{\partial\Omega_i\setminus\partial\Omega}$$

2 MATLAB Implementation for the Poisson's problem

First thing that was modified in the code is having a variable with the external elements for the Neumann, Robin and Dirichlet conditions with the external face number. The number of known and unknown degrees of freedom are changed accordingly.

```
         jon1=1;
         jon2=1;
         jon3=1;
for  i=1:64

         xco1=X(T(extFaces(i,1),1),1);
         yco1=X(T(extFaces(i,1),1),2);

         xco2=X(T(extFaces(i,1),2),1);
         yco2=X(T(extFaces(i,1),2),2);

         xco3=X(T(extFaces(i,1),3),1);
         yco3=X(T(extFaces(i,1),3),2);

         if (xco1==0 && xco2==0) || (xco1==0 && xco3==0) || (xco2==0 && xco3==0)
              extFacesN(jon1,1)=extFaces(i,1);
              extFacesN(jon1,2)=extFaces(i,2);
              jon1=jon1+1;
         elseif (yco1==1 && yco2==1) || (yco1==1 && yco3==1) || (yco2==1 && yco3==1)
              extFacesR(jon2,1)=extFaces(i,1);
              extFacesR(jon2,2)=extFaces(i,2);
              jon2=jon2+1;
         else
              extFacesD(jon3,1)=extFaces(i,1);
              extFacesD(jon3,2)=extFaces(i,2);
              jon3=jon3+1;
         end

end
```

Then, the matrices forming the stiffness matrix are then changed according to the new obtained weak forms.

3 Calculating boundary conditions

The boundary conditions were calculated from the given analytical solution of the problem. Feval function is used along with the previously calculated boundary nodes and their coordinates.

$$u_D = cos(exp(axy) + bcos(\pi(\gamma x + \kappa y^2))) \qquad on\ \Gamma_D$$

$$t = \kappa u_x = -\kappa sin(exp(axy) + bcos(\pi(\gamma x + \kappa y^2))) * (ayexp(axy) - b\pi\gamma sin(\pi(\gamma x + \kappa y^2))) \qquad on\ \Gamma_N$$

The Robin boundary condition is calculated similarly

$$g = \gamma u + \kappa u_y \qquad on\ \Gamma_R$$

4 Obtained result

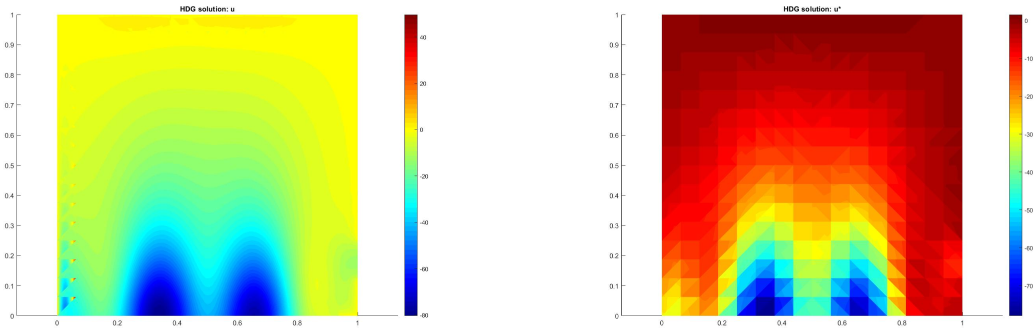

Figure 1: Pressure distribution and velocity contours at $t = 0.08$

Different meshes were used but the obtained results had a huge error. The code was revised several times but the error was not spotted.