UNIVERSITAT POLITÈNCICA DE CATALUNYA

HDG assignment #4

Rafel Perelló i Ribas

June 5, 2019

1 INTRODUCTION

In this assignment, the Poisson problem is aimed to be solved using a Hybridisable Discontinuous Galerkin (HDG) algorithm. To do that, it is started on the basis of a provided MATLAB code and it has been modified in order to take into account the possibility of imposing Neumann and Robin Boundary Conditions (BC). The domain in consideration is the following:

$$\begin{aligned}
\Omega &= [0,1]^2, \\
\Gamma_N &:= \left\{(x,y) \in \mathbb{R}^2 : y = 1\right\}, \\
\Gamma_R &:= \left\{(x,y) \in \mathbb{R}^2 : x = 1\right\}, \\
\Gamma_D &:= \partial\Omega \setminus (\Gamma_N \cup \Gamma_R).
\end{aligned}$$

It is aimed to solve the Poisson problem:

$$\begin{cases}
-\nabla \cdot (\kappa \nabla u) = s & \text{in } \Omega, \\
u = u_D & \text{on } \Gamma_D, \\
\boldsymbol{n} \cdot (\kappa \nabla u) = t & \text{on } \Gamma_N, \\
\boldsymbol{n} \cdot (\kappa \nabla u) + \gamma u = g & \text{on } \Gamma_R.
\end{cases}$$

2 HDG FORMULATION

The first step is to write the same equation element wise with the corresponding transmission conditions:

$$\begin{cases}
-\nabla \cdot (\kappa \nabla u_e) = s & \text{in } \Omega_e, \\
u_e = u_D & \text{on } \Gamma_D, \\
\boldsymbol{n} \cdot (\kappa \nabla u_e) = t & \text{on } \Gamma_N, \\
\boldsymbol{n} \cdot (\kappa \nabla u_e) + \gamma u = g & \text{on } \Gamma_R, \\
[\![u \cdot \boldsymbol{n}]\!] = \boldsymbol{0} & \text{on } \Gamma, \\
[\![\nabla u \cdot \boldsymbol{n}]\!] = 0 & \text{on } \Gamma.
\end{cases}$$

Now, the mixed variable **q** is introduced in the problem as $\boldsymbol{q} = -\kappa \nabla u$. Furthermore, the equation is separated into the local problem -with the PDEs and the Dirichlet BC- and the global problem -with the transmission conditions and the Neumann and Robin BC-. For that, the function $\hat{u}$ is introduced as a scalar defined only along $\Gamma \cup \Gamma_N \cup \Gamma_R$. It is an approximation of the trace of u in the inter-elemental faces as well as the Neumann and Robin BC edges.

$$\begin{cases} -\nabla\cdot\boldsymbol{q} = s & \text{in } \Omega_e,\\ \boldsymbol{q}+\kappa\nabla u = \boldsymbol{0} & \text{in } \Omega_e,\\ u_e = u_D & \text{on } \Gamma_D,\\ u = \hat{u} & \text{on } \partial\Omega_e\setminus\Gamma_D \end{cases}$$

$$\begin{cases} \boldsymbol{n}\cdot\boldsymbol{q} = t & \text{on } \Gamma_N,\\ \boldsymbol{n}\cdot\boldsymbol{q}+\gamma u = g & \text{on } \Gamma_R,\\ [\![\boldsymbol{q}\cdot\boldsymbol{n}]\!] = 0 & \text{on } \Gamma. \end{cases}$$

With this, the strong forms of the HDG have been derived. Note that the continuity of the primal variable is implicitly enforced with the definition of $\hat{u}$.
The weak form of the local problem consists in finding $(\boldsymbol{q_e}, u_e)$ such that:

$$\begin{aligned} (v,\nabla\cdot\boldsymbol{q_e})_{\Omega_e} &= (v,f)_{\Omega_e}\\ (\boldsymbol{w_e},\boldsymbol{q_e}+\nabla u) &= \boldsymbol{0} \end{aligned}$$

In the local problem there is not any difference from the pure Dirichlet problem so, integrating by parts, applying the divergence theorem and introducing the stabilisation term, the same weak form is obtained than in the Dirichlet problem:

$$\begin{aligned} < v,\tau_e u_e >_{\partial\Omega_e} + (v,\nabla\cdot\boldsymbol{q_e})_{\Omega_e} &= (v,f)_{\Omega_e} + < v,\tau_e u_D >_{\Gamma_D} + < v,\tau_e\hat{u} >_{\partial\Omega_e\setminus\Gamma_D}\\ (\nabla\cdot\boldsymbol{w},u_e)_{\Omega_e} - (\boldsymbol{w_e},\boldsymbol{q_e}) &= < \boldsymbol{n_e}\cdot\boldsymbol{w}, u_D >_{\Gamma_D} + < \boldsymbol{n_e}\cdot\boldsymbol{w},\hat{u} >_{\partial\Omega_e\setminus\Gamma_D} \end{aligned}$$

The global problem presents differences as the Neumann and Robin contributions must be added:

$$\sum_{i=1}^{n_el} < \mu, \boldsymbol{n_e}\cdot\hat{\boldsymbol{q}}_{\boldsymbol{e}} >_{\partial\Omega_e\setminus\partial\Omega} + < \mu, \boldsymbol{n_e}\cdot\hat{\boldsymbol{q}}_{\boldsymbol{e}} + t >_{\Gamma_N} + < \mu, \boldsymbol{n_e}\cdot\hat{\boldsymbol{q}}_{\boldsymbol{e}} + g - \gamma u >_{\Gamma_R} = 0$$

Substituting the numerical flux it is obtained weak form of the global problem:

$$\sum_{i=1}^{n_el} < \mu,\tau_e u_e >_{\partial\Omega_e\setminus\Gamma_D} + < \mu,\boldsymbol{n_e}\cdot\boldsymbol{q_e} >_{\partial\Omega_e\setminus\Gamma_D} - < \mu,\tau_e\hat{u} >_{\partial\Omega_e\setminus\Gamma_D} - < \mu,\gamma\hat{u} >_{\Gamma_N} = -\sum_{i=1}^{n_el} < \mu, t >_{\Gamma_N} + < \mu, g >_{\Gamma_R}$$

Three additional terms have appeared and their contribution has to be added to the stiffness matrix and force vector.

3 Code implementation

In the code, it has been implemented only the Robin BC type and the Neumann type has been considered a particular case of Robin BC where $\gamma = 0$. The implementation is summarised in the following steps:

1. Differentiate the external faces between faces belonging to Dirichlet and the Robin BC type:
   For that purpose, the same function *hdg_preprocess* has been used but then, an additional function has been defined to split the external faces into Dirichlet faces and Neumann faces. The information has been stored in the variable *infoFaces* with the same format.

2. Adapt the rows column elimination for Dirichlet BC:
   Now, there are degrees of freedom at some external edges. This means that nodes pertaining to Dirichlet and Robin BC are mixed. A function has been defined to differentiate between both.

3. Compute the contribution from the Robin BC:
   From the formulation of the weak form it is seen that a term has to be added to the stiffness matrix. This term is very similar to $A_{\hat{u}\hat{u}}$ with the difference that τ is replaced by γ and the integration is performed only along one face of the element, the one corresponding with the Robin BC. For that purpose, the same function that computes the elemental matrices has been used making some changes. In addition, the contribution to the body force has been computed in the same function. The force vector has the same form than the Neumann BC contribution in continuous Galerkin with the exception of the numerical integration. The implementation consists in a loop over each face corresponding to a Robin BC and at each iteration the contribution is assembled at the global stiffness matrix and force vector.

Finally, the convergence of the mixed variable $\boldsymbol{q}$ is required. For that, the same function to compute the $\mathscr{L}^2$ norm of the error of u has been used with some minimal changes.

4 Numerical example

In this assignment, to perform a numerical example, an analytical function has been considered and it has been calculated which functions should be used as source term and boundary conditions to get the same analytical function as a solution of the PDE. Then this solution is compared with the approximation from the HDG algorithm. The function is the following:

$$u(x,y) = -\log(\kappa x + ay) - \cos(\exp(bx + \gamma y))$$

The source term and functions corresponding to BC have been calculated:

$$s = -\nabla\cdot\left(\kappa\nabla u\right) = -\kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

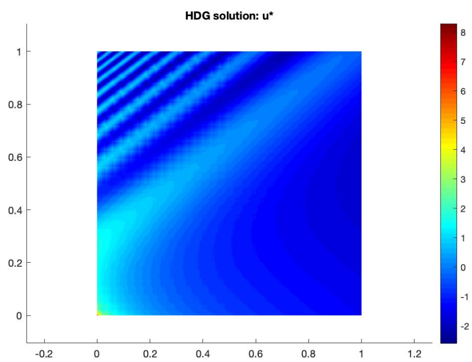
$$s = -\kappa\left(\frac{\kappa^2 + a^2}{(\kappa x + ay)^2} + \cos(\exp(bx+\gamma y))\cdot\exp(bx+\gamma y)^2\cdot(b^2+\gamma^2) + \sin(\exp(bx+\gamma y))\cdot\exp(bx+\gamma y)^2\cdot(b^2+\gamma^2)\right)$$

$$t = \boldsymbol{n}\cdot(\kappa\nabla u) = \kappa\cdot\frac{\partial u}{\partial y} = \kappa\left(\frac{-a}{\kappa x + ay} + \sin(\exp(bx+\gamma y))\cdot\exp(bx+\gamma y)\cdot\gamma\right)$$

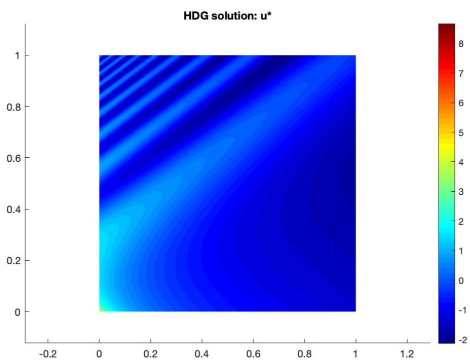
$$g = \boldsymbol{n}\cdot(\kappa\nabla u) + \gamma u = \kappa\cdot\frac{\partial u}{\partial x} + \gamma u = \kappa\left(\frac{-\kappa}{\kappa x + ay} + \sin(\exp(bx+\gamma y))\cdot\exp(bx+\gamma y)\cdot b\right) + \gamma\cdot\left(-\log(\kappa x + ay) - \cos(\exp(bx+\gamma y))\right)$$

5 Numerical result

For the numerical experiment, the values have been set as stated in the assignment: $a = 2, b = -3, \kappa = 1.5$ and $\gamma = 4$. The post-processed solution for polynomials of degree 4 is plotted along with the analytical solution:



(a) Post-processed solution



(b) Analytical solution

Although the computed solution is a good approximation of the analytical solution, the convergence plot does not behave as expected:

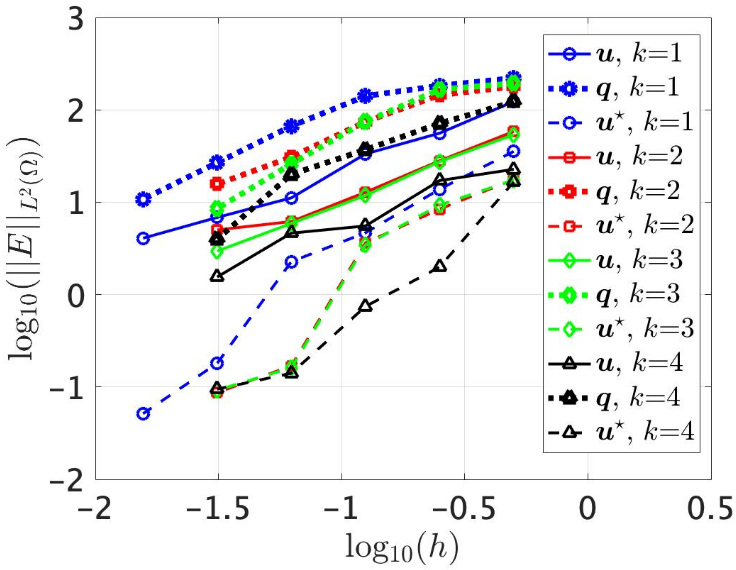
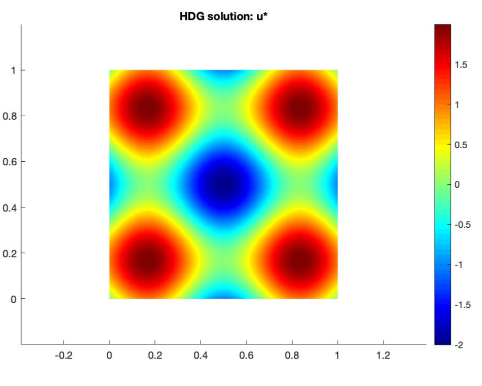


Figure 5.2: Convergence plot

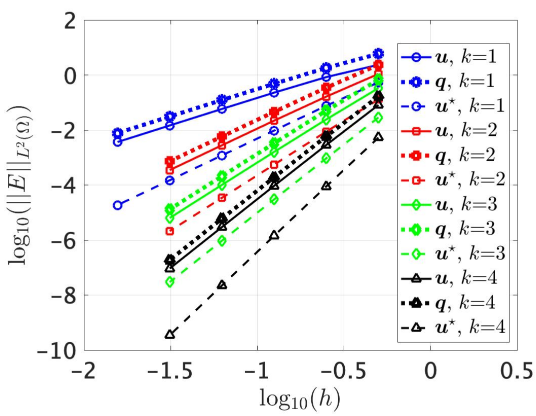
The convergence of most of the solutions is about of order 1. And although the post-processed solution present a lower error, its order of convergence is less than two in all cases. This is not what it was expected from the theory. The reason of this loss of order of convergence is that the analytical solution presents a singularity at the origin where the solution tends towards infinity due to the logarithmic term. It is well known that a singularity in the solution of a PDE reduces the order of convergence of the solution and that is what happens in this case.
To ensure that the algorithm has been applied correctly, a new analytical solution has been tried without any singularity. This function is:

$$u(x, y) = sin(3\pi \cdot x) + sin(3\pi \cdot y)$$

The same boundary conditions have been used. The source term and the functions of the BC term have been computed following the same steps than for the previous function.



(a) Post-processed solution for sinusoidal function



(b) Convergence plot for sinusoidal function

Now, it is seen that the order of convergence matches the theoretical one with an order of precision of 1%. The order of convergence for the primal and mixed variables is of order $p+1$ and the order of the post-processed variable is of order $p+2$.

6 Conclusions

In this assignment it has been implemented successfully the imposition of Robin boundary conditions (and with it also the Neumann type). The correctness of the implementation has been ensured by comparing the obtained convergence ratio with the expected one. It has also been checked that type of boundary condition does not affect the order of convergence of the solution.