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PROBLEM 2

$$\frac{\partial \underline{u}}{\partial t} - \nu \nabla^2 \underline{u} + (\underline{u} \cdot \nabla)\underline{u} + \sigma \underline{u} + \underline{\nabla p} = \underline{f} \quad \text{in } \Omega, \ t>0$$

$$\underline{\nabla} \cdot \underline{u} = 0 \quad \text{in } \Omega, \ t > 0$$

$$\underline{u} = 0 \quad \text{on } \partial\Omega, \ t > 0$$

$$\underline{u} = \underline{u_0} \quad \text{in } \Omega, \ t = 0$$

Only $Q_1 Q_1$ elements

a) Write the time - discrete problem using C-N

The Crank - Nicholson method is a semi - implicit method based on taking the average of the explicit and the implicit Euler method on the parabolic equation while enforcing the incompressibility condition.

$$\frac{\Delta u}{\Delta t} = \underline{u}_t^n + \theta \Delta \underline{u}_t = \underline{u}$$

The time discrete problem will be

$$\frac{u^{n+1} - u^{n}}{\Delta t} + (u^{n+1/2} \cdot \nabla) u^{n+1/2} - \nu \nabla u^{n+\frac{1}{2}} + \sigma u^{n+1/2} +$$

$$+ \nabla p^{n+1} = f^{n+1/2} \quad (1)$$

$$\text{div}(u^{n+1}) = 0$$

where

$$\Delta \tau = \tau^{n+1} - \tau^{n}$$

$$f^{n+\theta} = \theta f^{n+1} + (1-\theta) f^{n}$$

$$f^{n} \approx f(\tau^{n})$$

the pressure gradient has been evaluated at τ^{n+1},

together with the incompressibility condition

$$\left.\begin{aligned} u^{n+1/2} &= \frac{1}{2}\left(u^{n+1} + u^{n}\right) \\ f^{n+1/2} &= \frac{1}{2}\left(f^{n+1} + f^{n}\right) \end{aligned}\right\}$$

Each step enforces the solution of a modified stationary Navier-Stokes problem. It achieves second-order on the time step.

b) Derive the weak form obtained in a)

We have that $\Omega \in \mathbb{R}^2$

The weak form requires the introduction of classes of functions for the velocity and pressure fields. With respect to $\underline{u}$, the space of trial functions is denoted as S

$$S := \{ \underline{u} \in \mathcal{H}^1(\Omega) \mid \underline{u} = \underline{u}_D \text{ on } \Gamma_D \}$$

In our case $\underline{u} = 0$ on $\partial\Omega$

the weighting functions of the velocity, $\underline{w}$, belong to V. Functions in this class belong to the same space as S except that $\underline{w}$ vanish on Γ_D where the velocity is prescribed

Thus V is

$$V := \mathcal{H}^1_{\Gamma_D}(\Omega) = \{ \underline{w} \in \mathcal{H}^1(\Omega) \mid \underline{w} = 0 \text{ on } \Gamma_D \}$$

therefore we can say that for this problem $\underline{u}, \underline{v} \in V$

the space of functions (both trial and weighting) for the pressure is denoted as Q

$$Q := L_2(\Omega)$$

the weak form is obtained as

$$\int_\Omega u_t \cdot \underline{w} - \int_\Omega \nu \nabla^2 \underline{u} \cdot \underline{w} + \int_\Omega \left((\underline{u} \cdot \underline{\nabla}) \underline{u} \right) \cdot \underline{w} +$$

$$+ \int_\Omega \sigma \underline{u} \cdot \underline{w} + \int_\Omega \underline{\nabla p} \cdot \underline{w} = \int_\Omega \underline{f} \cdot \underline{w} \quad \forall \underline{w} \in V$$

$$\int_\Omega q \, \nabla \cdot \underline{u} = 0 \qquad \forall q \in Q$$

Now, integrating by parts the following terms

$$-\int_\Omega (\nu \nabla^2 \underline{u}) \cdot \underline{w} = \int_\Omega \nabla \underline{w} : \nu \nabla \underline{u} - \underbrace{\int_{\partial\Omega} \underline{w} \, \nu \nabla \underline{u} \cdot \underline{n} \, d\Gamma}_{0}$$

$$\int_\Omega \underline{\nabla p} \cdot \underline{w} = -\int_\Omega p \, \nabla \cdot \underline{w} + \underbrace{\int_{\partial\Omega} p \, \underline{w} \cdot \underline{n} \, d\Gamma}_{0}$$

therefore we reach: find $(\underline{u}, p) \in V, Q$

$(\underline{w}, u_t)$

$$\begin{cases} \delta \overset{+}{a}(\underline{w}, \underline{u}) + c(\underline{w}, \underline{u}, \underline{u}) - b(\underline{w}, p) + \sigma(\underline{w}, \underline{u}) = (\underline{w}, \underline{f}) \\ b(\underline{u}, q) = 0 \end{cases} \qquad (4)$$

with $u(0) = u_0$ and

$$a(\underline{w}, \underline{u}) := \int_\Omega (\nabla \underline{w}) : (\nu \nabla \underline{u}) \, d\Omega$$

$$b(\underline{u}, q) := \int_\Omega q \, \nabla \cdot \underline{u} \, d\Omega$$

$$c(\underline{w}, \underline{u}, \underline{a}) := \int_\Omega \underline{w} \cdot (\underline{a} \cdot \nabla) \underline{u} \, d\Omega$$

$$(\underline{w}, \underline{f}) = \int_\Omega \underline{w} \cdot \underline{f} \, d\Omega$$

$$\sigma(\underline{w}, \underline{u}) = \int_\Omega \sigma \underline{u} \cdot \underline{w} \, d\Omega$$

c) Discretise the weak form obtained in (b) and write the system of equations that has to be solved at each time step

We need to introduce local approximations for both the velocity components u_i^h and pressure p^h as for w_i^h and q^h. We then define the discrete spaces $V_h \in V$ and $Q_h \in Q$ and their associated Galerkin projection of the solution as

$$V_h = \text{span}\{N_1, \dots, N_{nv}\}$$

$$v_i^h = \sum_{j=1}^{nv} v_j N_j(x) \implies \text{Linear interpolati[on] for velocity}$$

$$Q_h = \text{span}\{\hat{N}_1, \dots, \hat{N}_{np}\}$$

$$p^h = \sum_{j=1}^{np} p_j \hat{N}_j(x) \implies \text{Linear interp. for pressure.}$$

where the auxiliary v belongs to the same space as the test function w^h, namely V^h

The Galerkin spatial discretization proceeds as:

for each $t \in]0, T[$, we define $u_D^h(t) \in V^h$ such that $u^h(t) = u(t) + u_D(t)$. Then, (J) we seek the auxiliary velocity field $\underline{v}^h(\cdot, t) \in V^h$ and pressure $p^h(\cdot, t) \in Q^h$, such that for all $(w^h, q^h) \in V^h, Q^h$,

$$\begin{cases} (\underline{w}^h, \underline{v}^h) + a(\underline{w}^h, \underline{v}^h) + c(\underline{v}^h; \underline{w}^h, \underline{v}^h) \\ + b(\underline{w}^h, p^h) + \sigma(\underline{w}^h, \underline{v}^h) = (\underline{w}^h, \underline{f}^h) \\ b(\underline{u}^h, q^h) = 0 \end{cases}$$

Important! the stability of the Galerkin method applied to the incompressible N-S equations depends on satisfying the LBB condition. As Q1 Q1 elements do not pass the LBB test, we have to add stabilization terms to each equation

To the first equation (momentum) it is added

$$\sum_{e=1}^{nel} \tau_{SUPG} \left((\underline{v}^h \cdot \nabla)\underline{w}^h, R(\underline{v}^h)\right)_{\Omega^e} +$$

$$+ \tau_{LSIC} (\nabla \cdot \underline{w}^h, \nabla \cdot \underline{v}^h)_{\Omega^e} = 0$$

to the second equation (pressure) it is added

$$\sum_{e=1}^{nel} \tau_{PSPG} (\nabla q^h, R(\underline{v}^h))_{\Omega^e} = 0$$

where $R(\underline{v}^h) = \underline{v}^h_t + (\underline{v}^h \cdot \nabla)\underline{v}^h - \nu \nabla^2 \underline{v}^h + \underline{\nabla p}^h - \underline{f}^h + \underline{\sigma v}^h$

Now if we consider the matrices

$$\underline{\underline{M}} = [\,\text{mat } N\,]^T [\,\text{mat } N\,] \Rightarrow M_{ij} = \int_\Omega N_i \cdot N_j$$

$$R_{ij} = \int_\Omega \sigma N_i N_j$$

$$K_{ij} = \int_\Omega \nabla N_j : \nu \nabla N_i$$

$$C_{ij} = \sum_{i,j=1}^{n_v} u_i u_j \, c(N_i, N_j, N_{k_1}) \quad k_1 = 1, n_v$$

$$G_{ij} = \hat{N}_i \, \nabla \cdot N_j$$

And the system is

$$\begin{cases} \underline{\underline{M}} \, \dot{\underline{v}}(t) + [\,\underline{\underline{K}} + \underline{\underline{C}}(\underline{u}(t)) + \underline{\underline{R}}\,]\,\underline{v}(t) + \underline{\underline{G}}\,\underline{p}(t) = \\ = \underline{f}(T, \underline{\alpha}(t)) \\ \underline{\underline{G}}^T \underline{v}(t) = \underline{0} \\ \underline{v}(0) = \underline{u}_0 - \underline{u}_0(0) \end{cases}$$

So, for the momentum equation

$$\underline{v}_t = \frac{\underline{f} - [\,\underline{\underline{K}} + \underline{\underline{C}}(\underline{v}) + \underline{\underline{R}}\,]\,\underline{v} - \underline{\underline{G}}\,p}{\underline{\underline{M}}}$$

Now, substituting in the θ - method scheme

$$\frac{\underline{v}^{n+1} - \underline{v}^{n}}{\Delta t} = \frac{1}{2}\left[\frac{\underline{f}^{n+1} - [\underline{\underline{K}} + \underline{\underline{C}}(\underline{v}) + \underline{\underline{R}}]\underline{v}^{n+1}}{\underline{\underline{M}}}\right]$$

$$- \frac{\underline{\underline{G}}\, p^{n+1}}{\underline{\underline{M}}} - \frac{1}{2}\left[\frac{\underline{f}^{n} - [\underline{\underline{K}} + \underline{\underline{C}}(\underline{v}) + \underline{\underline{R}}]\underline{v}^{n} - \underline{\underline{G}}\,p^{n}}{\underline{\underline{M}}}\right]$$

Yielding the system of equations

$$\begin{bmatrix} \underline{\underline{M}} + \frac{1}{2}\Delta t[\underline{\underline{K}} + \underline{\underline{C}}(\underline{v}) + \underline{\underline{R}}] & \frac{1}{2}\Delta t\, \underline{\underline{G}} \\ \underline{\underline{G}}^{T} & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\frac{1}{2}\Delta t\, f^{n+1} - \frac{1}{2}\Delta t\left[f^{n} - [\underline{\underline{K}} + \underline{\underline{C}}(\underline{v}) + \underline{\underline{R}}]v^{n} + \underline{\underline{G}}\, p^{n}\right]}{\underline{\underline{M}}} \\ \underline{0} \end{bmatrix}$$

d) Propose an algorithm to solve at each time-step

If we approximate the convective term implicitly

$$\underline{\underline{C}}(\underline{v}) = [(\underline{v}\cdot\nabla)\underline{v}]^{n+1/2}$$

then we have to use Newton-Raphson algorithm

The problem will thus be, considering an initial guess $(\underline{u}^0, p^0)$ at each Newton sub-iter- to look for $(\underline{u}^{n+1}, p^{n+1})$ such that

$$\underline{\underline{J}}(\underline{u}^n, p^n)\left[(\underline{u}^{n+1}, p^{n+1}) - (\underline{u}^n, p^n)\right] =$$

$$= R(\underline{u}^n, p^n)$$

And we want to achieve $R(\underline{x}) = \underline{0}$

The Jacobian matrix will be computed as

$$[\underline{\underline{J}}(\underline{x})]_{ij} = \frac{\partial r_i}{\partial x_j}(\underline{x})$$

$$\underline{\underline{J}} = \begin{bmatrix} \frac{\partial r_1}{\partial \underline{u}} & \frac{\partial r_1}{\partial p} \\ \frac{\partial r_2}{\partial \underline{u}} & \frac{\partial r_2}{\partial p} \end{bmatrix} \quad \text{with } \underline{r} = \underline{\underline{A}}\,\underline{x} - \underline{b}$$

Therefore at each iteration we will solve the linear system of equations

$$\underline{\underline{J}}(\underline{x}^{n})\,\Delta\underline{x}^{n+1} = -\underline{r}(\underline{x}^n)$$

and update the solution as

$$\underline{x}^{n+1} = \underline{x}^n + \Delta\underline{x}^{n+1}$$

e) Are the methods behaving as expected

Yes, They are behaving as should be expected.

The fact That the Reynolds is high means that the non-linear and non-symmetric Terms associated to the convective matrix in the momentum equation gain weight. This is one of the main issues in incomp. flows. High Re number flows are convection-dominated, and stabilization techniques must be used.

Even so, if the initial guess which is used for the Newton-R scheme is not sufficiently close to the final solution, the Newton scheme diverges, and not The Picard, which is more robust.

~~If~~ However, The Picard method would eventually converge if Re is increasing

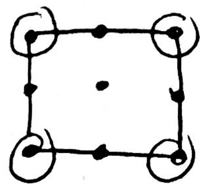
As a solution, eventually, I would suggest Taking as initial guess the solution for Re = 500 for instance, or using numerical continuation Techniques.

PROBLEM 1

a) Is the pair of finite element spaces suitable to discretise the equation ?

If we are in the context of a continuous Galerkin method, using a Taylor - Hood element is enough to pass the LBB test and it is not necessary to use stabilization techniques

A continuous biquadratic velocity and continuous bilinear pressure satisfy the LBB condition and has quadratic convergence



$Q_2 Q_1$,

The problem is that with this configuration the pressure and velocity are interpolated at different nodes, and some people prefer to use $Q_1 Q_1$ elements and stabilization techniques.

b) Consider a stabilised GLS formulation for the Stokes problem. Is this a suitable method for the problem at hand?

It is not needed, since the use of Q_2 Q_1 elements already guarantee the satisfaction of the LBB condition and therefore the numerical imposition of incompressibility is satisfied.

Therefore it could be a suitable method but not for the problem at hand.

c) How is the pressure approximated in order to fulfill the LBB condition? Which is the global size of the resulting DG problem?

The pressure has to be approximated with a polynomial of degree ~~5~~ 5 in order to satisfy the LBB condition, that is, 6 points.

Since all the degrees of freedom of the elements are connected to the dofs of other elements only through the edges, we will have

12 nodes of pressure for each edge
There are 8 edges

$12 \cdot 8$ pressure unknowns $\rightarrow$ 96 unk.

14 nodes of velocity for each edge
28 degrees of freedom for each edge (velocity)
8 edges

$28 \cdot 8 = 224$ dofs for velocity
In total $\rightarrow$ $224 + 96 = 320$

d) HDG

For HDG we have two types of unknown

- u, q $\rightarrow$ local unknowns, solved only local
- $\hat{u}$ $\rightarrow$ Boundary conditions, unique for both

elements in contact to solve the problem

The global problem is only concerned with $\hat{u}$ and has unknowns only on the edges

Now the unknowns for pressure are duplicated on the vertices
8 vertices $\rightarrow$ 16 unknowns
Equal for velocity $\rightarrow$ 224 dofs

As for the local variables

- velocities along the boundaries
- one constant value for each element for pressure as the average on the boundaries

e) Sketch the algorithm

Load the mesh
Set Reference element

↓

Simulation parameters

↓

Assemble and solve global problem

↓

Loop on elements

↓

Solve local problem

↓

Element loop finished?

No

Yes ↓

Loop on elements

↓

Post-process solution

↓

Element loop finished?

No

Yes ↓

Compute error

Loop on elements

↓

Element integrals

↓

Loop on elem. faces

↓

Face integrals