 ADRIÀ GALOFRÉ AISA

FINAL EXAM

Problem 1:

Consider:

$$\begin{cases} u_t + a u_x - \nu u_{xx} = 0 & x \in (0,1), \quad t > 0 \\ u(x,0) = u_0(x) & x \in (0,1) \\ u_x(0) = u_x(1) = 0 \end{cases}$$

(1) Discretize the BVP in time using Lax-Wendroff.

Since it is not possible to design one-stage schemes more than second order accurate schemes for convection-diffusion problems (due to C^0 and 2nd derivatives incompatibility) a two-stage scheme is proposed.

Lax-Wendroff ≡ Explicit Padé scheme $R_{2,0}$

using the approximation:

$$u(t^{n+1}) = u(t^n) + \Delta t \frac{\partial}{\partial t}\left(u + \frac{\Delta t}{2}\frac{\partial u}{\partial t}\right)\Big|_{t=t^n} + \mathcal{O}(\Delta t^3)$$

and the 2 stages:

$$\boxed{\begin{aligned} u^{n+1/2} &= u^n + \tfrac{1}{2}\Delta t\, u_t^n \\ u^{n+1} &= u^n + \Delta t\, u_t^{n+1/2} \end{aligned}}$$

(2) Weak-form:

Let us write one stage discretization

$$\frac{u^{n+\alpha} - u^n}{\alpha \Delta t} = u_t^{n+\beta} = -a\nabla u^{n+\beta} + \nu \nabla^2 u^{n+\beta}$$

and then multiply each term by $w \in H_0^1(\Omega)$ test function and operating, we get the weak form associated:

$$\left(\frac{u^{n+\alpha}-u^{n}}{\alpha\Delta t}, w\right) + a(u^{n+\beta}, w) = 0.$$

where a:

$$a(u^{n+\beta}, w) = (a\cdot\nabla u^{n+\beta}, w) + (\nu\nabla u^{n+\beta}, \nabla w).$$

(3) Discretize in space using Galerkin finite element method.

The spaces used to solve the weak form are

$S_t := \{ u \mid u(\cdot,t) \in H^1(\Omega),\ t \in [0,T]$ and $u(x,t) = u_D$ for $x \in \Gamma_D \}$.

+

the space of weighting functions:

$V := \{ w \in H^1(\Omega) \}$.

if we want to use the Galerkin discretization we must define S^h and V^h such that:

$$S_t^h := \{ u \mid u(\cdot,t) \in H^1(\Omega),\ u(\cdot,t)|_{\Omega^e} \in \mathcal{P}_p(\Omega^e)\ t \in [0,T]\ \forall e \text{ and } u = u_D \text{ on } \Gamma_D \}$$

$$V^h := \{ w \in H^1(\Omega) \mid w|_{\Omega^e} \in \mathcal{P}_p(\Omega^e)\ \forall e \text{ and } w = 0 \text{ on } \Gamma_D \}$$

$$\begin{cases} \left(\dfrac{u^{h,n+\alpha}-u^{h,n}}{\alpha\Delta t}, w^h\right) + a(u^{h,n+\beta}, w^h) = 0 \\ (w^h, u^h(x,0)) = (w^h, u_0(x)) \end{cases}$$

with:

$$u^h(x,t) = \sum_{i\in\eta\setminus\eta_D} N_i^{(x)} u_i(t) + \sum_{i\in\eta_D} N_i(x)\cdot u_D(x_i,t).$$

with the discrete form.

$$\boxed{M u_t + (C+K)u = 0}$$

with M, C and K of the form:

$$\boxed{\begin{aligned} M^e_{ij} &= \int_{\Omega^e} N_i N_j \, d\Omega \\ C^e_{ij} &= \int_{\Omega^e} N_i \, (a \cdot \nabla N_j) \, d\Omega \\ K^e_{ij} &= \int_{\Omega^e} \nabla N_i \cdot (\nu \nabla N_j) \, d\Omega \end{aligned}}$$

4) Solutions for (a, ν_i) with $(a, \nu_i) = (1; 0{,}01, 0{,}01, 0{,}1)$, $h = 0.1$ and $\Delta t = (0.1; 0{,}05; 0{,}1)$ presents numerical ~~os~~ oscillations on the solution. Why?

The numerical oscillations are due to methods like the one derived, Galerkin + explicit Padé, are conditionally stable.

If a Fourier analysis is done and the amplification factor calculated, it is of the form:

$$G(\xi, C, d, c, r) = R_{n1q,0}(z) \quad \text{with} \quad z := \frac{K(\xi, d) + A(\xi, C) - rM(\xi)}{M(\xi)}$$

and the Courant and Peclet number will play a major role on its stability.

It can be seen that since the convection ~~term~~ parameter dominates over the diffusion Galerkin formulation lacks sufficient spatial stability when boundary layers appears.

FINITE ELEMENTS IN FLUIDS

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Problem 2:

Consider the Stokes problem:

$$\begin{cases} -\nu \nabla^2 u + \nabla p = b & \text{in } \Omega \\ \nabla \cdot v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

(1) Derive the weak form. Detail the steps.

To ~~derive~~ derive the weak form, we firstly need to define the next spaces:

- Trial solution space:

$$S := \{ v \in H^1(\Omega) \mid v = v_D \text{ on } \Gamma_D \}$$

- Weighting functions of the velocity, w.

$$V := \{ w \in H^1(\Omega) \mid w = 0 \text{ on } \Gamma_D \}$$

- Space for the pressure.

$$Q = \mathcal{L}_2(\Omega)$$

Also let us define the boundary tractions in order to have a well posed BVP:

$$-pn + \nu (n \cdot \nabla) v = t \quad \text{on } \Gamma_N \quad (\text{Neumann b.c.})$$

also called "pseudo-traction"

So applying the divergence term ~~to~~ the pressure gradient term and to the viscous term, the weak form reads:

find $(v,p) \in S \times Q$ such that

$$a(w,v) + b(w,p) + b(v,q) = (w,b) + (w,t)_{\Gamma_N} \quad \forall (w,q) \in V \times Q$$

since $v_D = 0$ (Dirichlet BC)

with

$$a(w,v) = \int_\Omega \nabla w : \nu \nabla v \, d\Omega.$$

2. Discretize the problem using Galerkin finite element approximation

we need to introduce approximations for velocity and pressure:

$$v_i^h \in S^h$$

$$p^h \in Q^h$$

and weighting function:

$$w^h \in V^h.$$

and then:

$$\boxed{a(w^h, v^h) + b(w^h, p^h) = (w^h, b^h) + (w^h, t^h)_{\Gamma_N} \quad , b(u^h, q^h) = 0}$$

with $V = 0$ on $\partial\Omega$ and:

$$u_i^h = \sum_{A \in \eta / \eta_{D_i}} N_i(x)\, u_i$$

$$p^h(x) = \sum_{\hat{A} \in \hat{\eta}} \hat{N}_i(x)\, p_i$$

with matrix form of the problem:

$$\boxed{\begin{bmatrix} K & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}}$$

with

K	$a(w, u)$	$\int \nabla w : \nu \nabla v \, d\Omega.$
G	$b(w, p)$	$-\int p \nabla w \, d\Omega.$
f	$(w, b) + (w, t)_{\Gamma_N}$	$\int w b \, d\Omega + \int_{\Gamma_N} w t \, d\Omega$
G^T	$b(u, q)$	$-\int_\Omega q \nabla u \, d\Omega.$

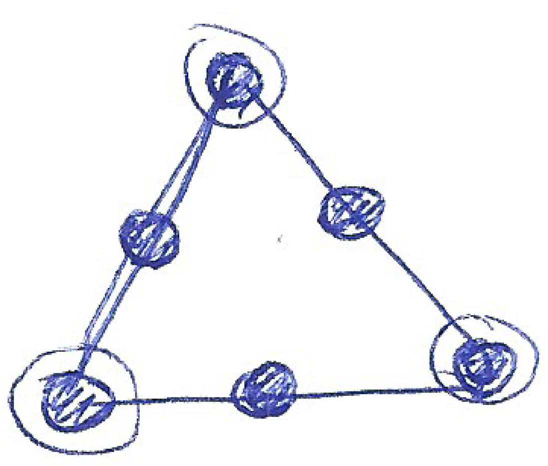
(3) Discuss whether the problem can be solved using $P_1 P_1$ and $P_2 P_1$ elements.

In order to know if the pairs proposed are appropiate to solve the problem, they must fullfil the inf-sup condition

the inf-sup condition states that the velocity and pressure spaces have to be linked, so the pair choosed have to fullfil:

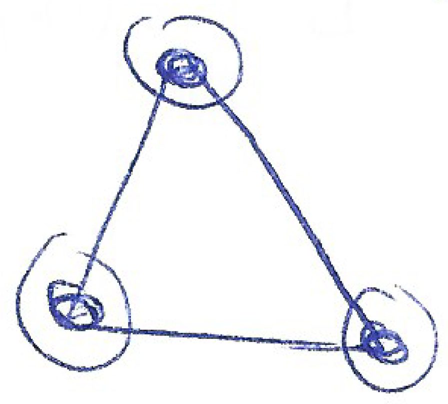
$$\inf_{q^h \in Q^h} \sup_{w \in V^h} \frac{(q^h, \nabla w^h)}{\|q\|_0 \, \|w^h\|_1} \geq \alpha > 0$$

This condition is fullfilled for the pair P_2/P_1 (quadratic velocity / linear pressure)



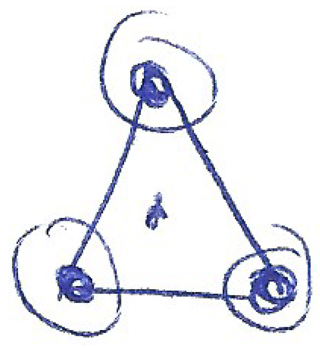
O Presure
● velocity

whereas the P_1/P_1 method will not fullfil this condition, and won't give proper results.



O pressure
● velocity

This element can be improved using bubble functions in the velocity nodes and then fullfil the inf-sup condition:



P_1^+ / P_1

O pressure
● velocity
▲ Bubble.

④ Derive a stabilized GLS for the Stokes formulation. Use linear elements for both velocity and pressure

To derive the stabilized GLS for Stokes problem lets ~~us~~ write the problem in the variational form.

$$a(w,v)+b(w,p)+b(v,q) = (w,b) \quad , \forall (w,q) \in V \times Q$$

and define the square of the residual of the momentum eq:

$$\mathcal{L}_S(v,p) := (-\nu\nabla^2 v + \nabla p - b, -\nu\nabla^2 v + \nabla p - b)$$

stationary condition $\mathcal{L}_S$ implies:

$$\left.\frac{d\mathcal{L}_S(v+\varepsilon w, p+\varepsilon q)}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

that implies:

$$\begin{cases} (-\nu\nabla^2 w, -\nu\nabla^2 v + \nabla p - b) = 0 & \forall w \in V \\ (\nabla q, -\nu\nabla^2 u + \nabla p - b) = 0 & \forall q \in Q \end{cases}$$

the stabilized discrete problem (using the Galerkin interpolation) is now:

$$\begin{cases} a(w^h, v^h) + b(w^h, p^h) + \sum_{e=1}^{nel} \tau_e(-\nu\nabla^2 w^h, -\nu\nabla^2 v^h + \nabla p^h - b^h)_{\Omega_e} = (w^h, b) + (w^h, t^h)_{\Gamma_N} \\ b(v^h, q^h) - \sum_{e=1}^{nel} \tau_e(\nabla q^h, -\nu\nabla^2 v^h + \nabla p^h - b^h)_{\Omega_e} = 0 \end{cases}$$

with τ_e stabilization parameter of the form:

$$\boxed{\tau_e = \alpha_0 \frac{h_e^2}{4\nu}}$$

For linear elements: $\nabla^2 = 0$ and $\alpha_0 = 1/3$

$$\boxed{\begin{cases} a(w^h, v^h) + b(w^h, p^h) = (w^h, b^h) + (w^h, t^h)_{\Gamma_N} \\ b(v^h, q^h) - \sum_{e=1}^{nel} \tau_e(\nabla q^h, \nabla p^h) + \sum_{e=1}^{nel} \tau_e(\nabla q^h, b^h)_{\Omega_e} \end{cases}}$$

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Problem 3:

1. Consider 1D steady-state convection diffusion problem:

$$\begin{cases} u_x - \nu u_{xx} = f(x) & \text{for } 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

we consider $\nu = 1$ and $\nu = 10^{-2}$

a). How the Galerkin method will behave if a mesh of 10 uniform linear elements is used to approximate the exact solution

1. First we multiply the terms in the strong form by test function $w \in V$, $V := \{ w \in H^1(\Omega) \mid w = 0 \text{ on } \Gamma_D \}$ and integrate by parts:

$$\int_0^1 (w\, u_x + w_x \nu u_x)\, dx = \int_0^1 w\, f(x)\, dx$$

2. Discretize the weak form: $h = 0.1$ $\left(n_{el} = 10 \;/\; h = \frac{L}{n_{el}} = \frac{1}{10}\right)$

$$\int_0^1 \sum \left(N_i \frac{\partial N_j}{\partial x} + \nu \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \right) u\, dx = \int_0^1 N_i f(x)\, dx$$

3. In matrix form, matrix are:

$$C^e = \int_\Omega \begin{pmatrix} N_1 \frac{\partial N_1}{\partial x} & N_1 \frac{\partial N_2}{\partial x} \\ N_2 \frac{\partial N_1}{\partial x} & N_2 \frac{\partial N_2}{\partial x} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & +1 \\ -1 & +1 \end{pmatrix}$$

$$K^e = \nu \int_\Omega \begin{pmatrix} \frac{\partial N_1}{\partial x} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial x} \frac{\partial N_2}{\partial x} \\ \frac{\partial N_2}{\partial x} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} \frac{\partial N_2}{\partial x} \end{pmatrix} dx = \frac{\nu}{0{,}1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

with $N_1(\xi) = \frac{1}{2}(1-\xi)$, $N_2(\xi) = \frac{1}{2}(1+\xi)$

4. Galerkin equation at an interior node j:

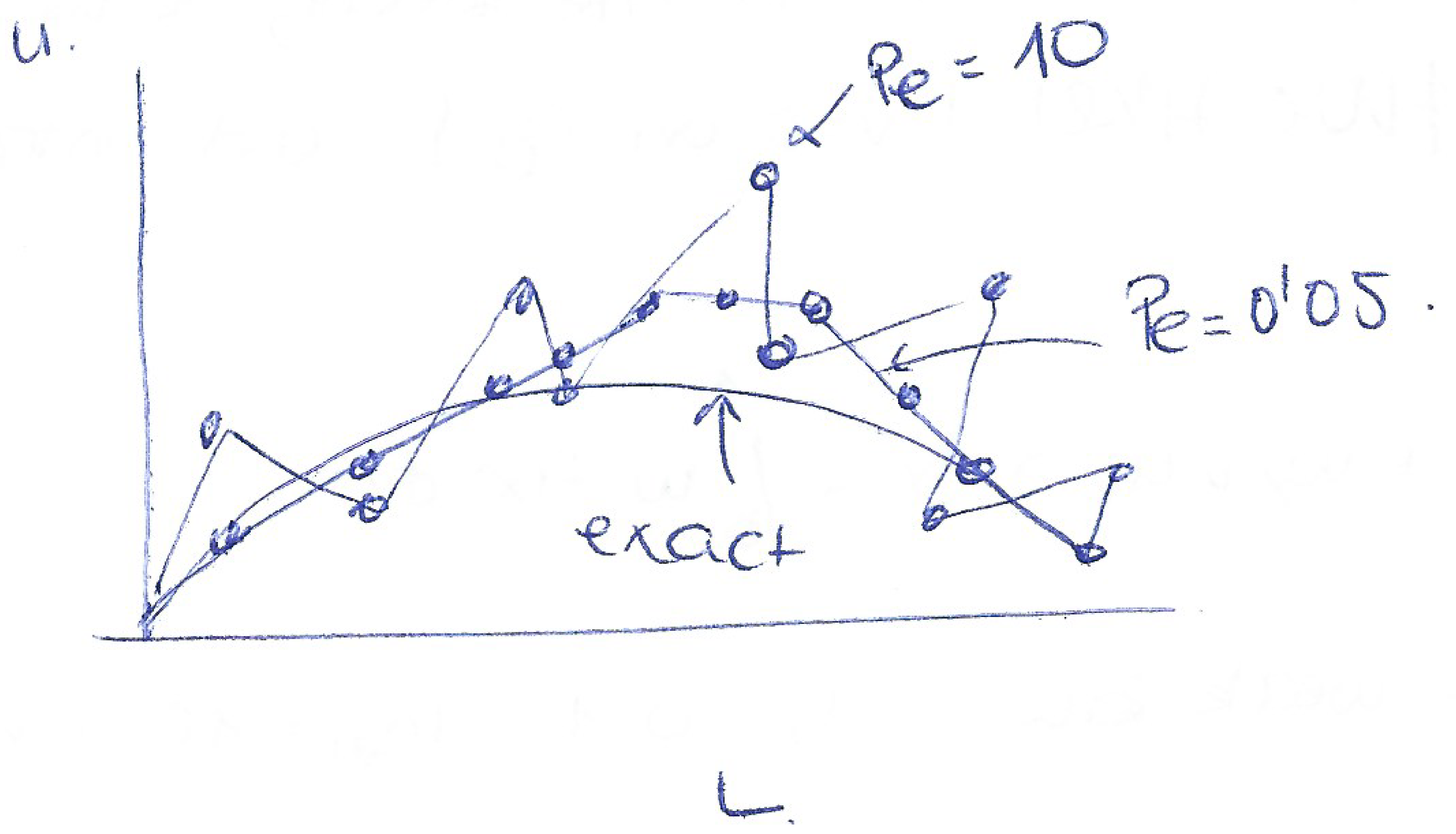
$$\left(\frac{u_{j+1}-u_{j-1}}{0.2}\right)-\nu\left(\frac{u_{j+1}-2u_j+u_{j-1}}{(0,1)^2}\right)=RHS.$$

5. If we introduce the Peclet number $Pe=\frac{ah}{2\nu}$.

$$\frac{1}{0{,}2}\left(\frac{Pe-1}{Pe}u_{j+1}+\frac{2}{Pe}u_j-\frac{Pe+1}{Pe}u_{j-1}\right)=RHS.$$

It can be observed that for $Pe>1$ we will obtain node-to-node oscillations in the solution. For the parameters in the problem.

$\nu=1.$	$\nu=0{,}01.$
$Pe=\frac{0{,}1}{2}=0'05$	$Pe=\frac{0{,}1}{0{,}01}=10.$



b) GLS method. τ?

1. Let's define the operator.

$$P(w)=\mathcal{L}(w)=\underbrace{a\nabla w}_{SUPG}-\overbrace{\nabla(\nu\nabla w)}^{0\text{ fo L.F.F}}+\underbrace{\sigma w}_{Galerkin}.$$

2. weak form GLS:

$$\boxed{a(w^h,u^h)+C(a;w^h,u^h)+\sum_e\int_{\Omega^e}\mathcal{L}(w^h)\tau\left[\mathcal{L}(u^h)-f(x)\right]d\Omega+\sigma w=(w^h,f(x))+(w^h,h)_{\Gamma_N}}$$

3. τ? For linear elements according to the slides.

$$\boxed{\tau=\frac{h}{2a}\left(\coth(Pe)-\frac{1}{Pe}\right)}$$

Since it is consistent and symmetric will improve the results, but it still 'carries' the Galerkin inestabilities: ~~until~~ $\underline{\sigma w}$. For Linear Finite Elements we will get the same result as for SUPG but the Galerking term weighted $\underline{1+\sigma\tau}$ times.

(2) Consider a 1D pure convection equation:

$$u_t + a u_x = 0 \quad \text{for} \quad 0 < x < 1, \quad t > 0$$

$$u(x,0) = f(x) \quad \text{for} \quad 0 < x < 1.$$

a) Comment on the results.

We can see that for LW and CN with a Courant number equals to 0.5 the results are very similar for both methods. But when $Cu = 0.9$, while CN keeps providing a good approximation LW becomes inestable.

If a Fourier analysis for stability is done, the amplifications factors for both methods are:

$$|G_{CN}| = 1. \qquad |G_{LW}| = \left| \frac{1-(\frac{2}{3}+2C^2)\sin^2\frac{\xi}{2} - iC\sin\xi}{1-\frac{2}{3}\sin^2\frac{\xi}{2}} \right|$$

So, CN is unconditionally stable whereas LW is not, its stability will depends on the courant number. Actually it will be stable for $C < 0{,}57$

b). it is better to solve the problem using LW with a lumped mass matrix?

If a diagonal mass matrix is used, the amplification factor for LW will become:

$$G_{LW_{LM}} = 1 - 2C^2 \sin^2 \frac{\xi}{2} \; i \, C \sin \xi .$$

which is stable up to $C^2 \leq 1$..

So with the $C = 0'9$. we will still have a stable solution.

③ Picard and Newton-Raphson methods on Navier-Stokes

Regarding the solution for $Re = 100$, we see that both methods behave, and converge, as expected, that is, linearly for Picard and Quadratic for N-R.

When increasing the Reynolds number to $Re = 1000$ we see that Picard methods continue converging linearly ~~[illegible]~~ while N-R does not converge

④ Advantages HDG

Due to the hybridization it is capable to reduce significantly the Dofs and get faster results (in time), faster convergence ~~[illegible]~~ for small degree polynomials.