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Final Exam

Problem 1

a) For the given equation, $Q_2 Q_1$ elements are suitable for the discretization because it is LBB stable. The Q_2 elements take care of the convective term which has a double derivative, unlike the Q_1 element for pressure, linear elements are enough to get accurate results.

~~b) For the problem at hand, GLS stabilisation is suitable, but some conditions should be taken into account for guaranteed convergence:~~

$$\cancel{\tau_1 = \alpha_0 \frac{h^2}{4\nu} \quad \text{and} \quad \tau_2 = 0}.$$

b) For this problem, having $Q_2 Q_1$ elements which are LBB stable, stabilization techniques are ~~unecessec~~ unecessary, therefore the GLS method is not suitable.

c) To fulfill the LBB condition, first the pressure should be approximated for a degree smaller than 6. Optimal convergence is obtained when the ~~power~~ degree of u is bigger than the degree of p by 1, therefore p should have an apporoximation of degree 5.

~~3~~ e) After loading the mesh, and problem parameters, the assembly of the global matrix is required. To do so, for each element, the integrals is computed and assembled in the global matrix. The same goes for f. After assembling the matrices, the matrix equation is solved, and the ~~post~~ solution is calculated. At the end, ~~er~~ the results are compared to analytical results.

Problem 2

a) $\frac{\partial u}{\partial t} - \vartheta \nabla^2 u + (u \cdot \nabla) u + \sigma u + \nabla p = f$

From the slides of incompressible flow, using Crank-Nicolson we get

$$\begin{cases} \frac{u^{n+1} - u^n}{\Delta t} + \vartheta \nabla^2 u^{n+\frac{1}{2}} + \left((u \cdot \nabla) u\right)^{n+\frac{1}{2}} + \sigma u^{n+\frac{1}{2}} + \nabla p^{n+\frac{1}{2}} = f^{n+\frac{1}{2}} \\ \nabla \cdot u^{n+1} = 0 \end{cases}$$

b) integrating each term, and multiplying by a test function w

$$\int_\Omega w \cdot \frac{u^{n+1} - u^n}{\Delta t} \, d\Omega + \int_\Omega w \cdot \vartheta \nabla^2 u^{n+\frac{1}{2}} \, d\Omega + \int_\Omega w \cdot \left((u \cdot \nabla) u\right)^{n+\frac{1}{2}} d\Omega$$

$$+ \int_\Omega w \cdot \sigma \cdot u^{n+\frac{1}{2}} \, d\Omega + \int_\Omega w \cdot \nabla p^{n+\frac{1}{2}} \, d\Omega = \int_\Omega w \cdot f^{n+\frac{1}{2}} \, d\Omega$$

Doing an integration by parts for the 2nd term and the term with pressure:

$$\begin{cases} \int_\Omega w \cdot \frac{u^{n+1} - u^n}{\Delta t} \, d\Omega + \int_\Omega (\nabla w) : (\vartheta \nabla u)^{n+\frac{1}{2}} d\Omega + \int_\Omega w \cdot \left((u \cdot \nabla) u\right)^{n+\frac{1}{2}} d\Omega \\ + \int_\Omega w \cdot \sigma \cdot u^{n+\frac{1}{2}} d\Omega + \int_\Omega (\nabla \cdot w) \, p^{n+1} \, d\Omega = \int_\Omega w \cdot f^{n+\frac{1}{2}} \, d\Omega \\ \int_\Omega (\nabla \cdot q) \, u \, d\Omega = 0 \end{cases}$$

for the boundary condition, $u = 0$ therefore no extra terms will be added to the right hand side.

c) solving for the steady part of the problem,

~~the equation~~ the operation on terms of the 2nd equation and the pressure term of the 1st equation are the same, therefore the operation will be considered the same: $G = \int_{\Omega} \nabla w \, d\Omega$ for the same shape function.

For the terms multiplying $u^{n+\frac{1}{2}}$, they will be considered as

$K + C(v) + S$

Therefore the matrix equation will be:

$$\begin{pmatrix} K + C(v) + S & G^T \\ G & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

(for one time step)

while $F = \int_{\Omega} w.f \, d\Omega$.

~~For the given time steps:~~

$$\cancel{\begin{pmatrix} K + C(v) + S & G^T \\ G & 0 \end{pmatrix} \begin{pmatrix} u^{n+\frac{1}{2}} \end{pmatrix}}$$

Having the unsteady term: $M\left(\frac{u^{n+1} - u^n}{\Delta t}\right)$

we get: $A = M + \Delta t(K + C + S)$

Therefore the given equation will have this form:

$$\begin{pmatrix} A & G^T \\ G & 0 \end{pmatrix} \begin{pmatrix} u^{n+\frac{1}{2}} \\ p^{n+1} \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{M}{\Delta t} u^n \\ 0 \end{pmatrix}$$

d) Having time steps in between the steps, a quadratic solver is prefered, which is the Newton-Raphson method.

e) After setting $\sigma = 0$, the equation will be similar to the one given in the slides.

Looking at the results, both methods are behaving as expected.
From the plot of $Re = 100$, both methods converge and as expected, Newton-Raphson converges quadratically, and Picard method converges linearly.

When the Reynolds number increases ($Re = 1000$), instabilities occure, that will lead to different wrong answers ~~for~~ in linear elements, and this is what happens in the Newton-Raphson method. Although the Picard method is more time consuming, for this ~~Re~~ Re it will still be stable and converge to the answer.