Optional Problem - FEF

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1 Steady Coupling

1.1 Problem Description

The problem presented consists in a coupling between the Stokes and Transport equations. The coupling appears in Stokes equation with the viscosity dependent on density and in the transport equation with the convective and source terms dependent on velocity. This problem can be understood as pollutants (ρ) being carried by a very viscous flow (u). The parameters used in this problem will be the same as were used in questions 1 and 3.

For the Transport equation, the domain of $(2, 3)$ is chosen with the boundary conditions $\rho = 0$ in Γ_2 and $\rho = 1$ in Γ_4. The initial viscosity parameter is $\nu = 10^{-8}$, the diffusivity is $\mu = 10^{-8}$ and no source term was considered. For the Stokes equation, the cavity configuration was chosen, with the same domain as the transport problem but with the boundary conditions $v_y = -1$ in Γ_2.

1.2 Implementation

In order to solve this problem, 2 strategies were considered. The first one was to solve the problem in a monolithic way, computing the residuals and derivatives of the residuals of each equation and solving the problem simultaneity. Due to consistency problems with the computation of pressure, this strategy was discarded.

The second strategy was to solve the problem iteratively in a "brute force" manner. It consists in solving the equations separately and recomputing with the new values until the residuals are small enough. The algorithm is described in Figure 1.1. First a initial guess of velocity is made and with that the transport equation is computed using the guessed velocity on the convective term. With the values of density, the viscosity can be calculated and with that the Stokes equation. Now with $\mathbf{v}$ and ρ, the residuals of each equation can be computed as $R_\rho = |\rho_{k+1} - \rho_k|$ and $R_\mathbf{v} = |\mathbf{v}_{k+1} - \mathbf{v}_k|$. If the residuals are small enough, the calculation is complete, if not, the algorithm restarts with the new values of velocity and density until convergence.

The Q2Q1 element for velocity and pressure was chosen for this implementation together with quadratic quadrilaterals for density. This facilitates the implementation because, with this set up, the number of nodes and gauss points of velocity and density are the same, making it easier to make the evaluation of each variable on the integration points.

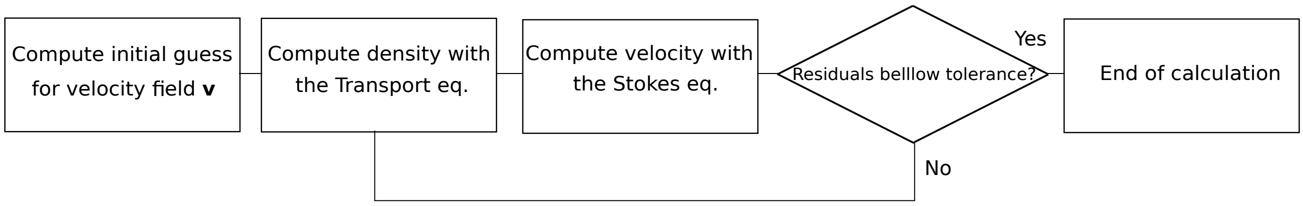


Figure 1.1: Iterative algorithm to solve the coupled problem.

1.3 Results

In order to obtain reasonable results, a space discretization of $h = 0.1$ (same as the one used on the previous report) was chosen. Also, the tolerance of the iterative scheme was chosen to be 10^{-4} for both residuals. A SUPG stabilization scheme was used for the transport equation in order to avoid oscillations. The convergence analysis of the problem can be seem in Figure 1.2. It can be observed that only 18 iterations was required to converge both equations. It is important to notice that the Stokes equations converge significantly earlier then the Transport equation, which may be due to the fact that the density field is highly affected by the velocity values trough the convective term.

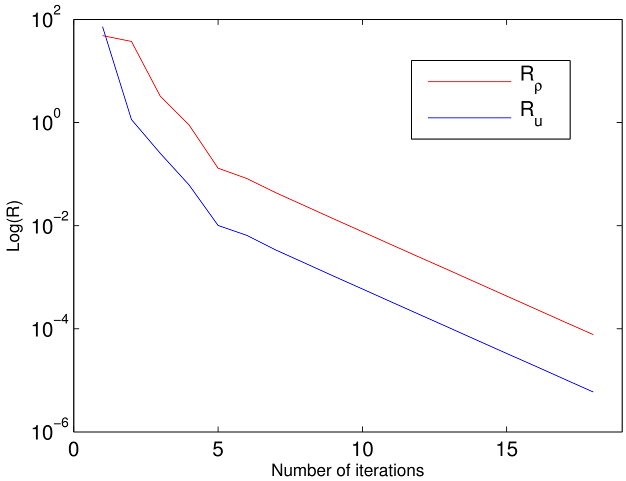
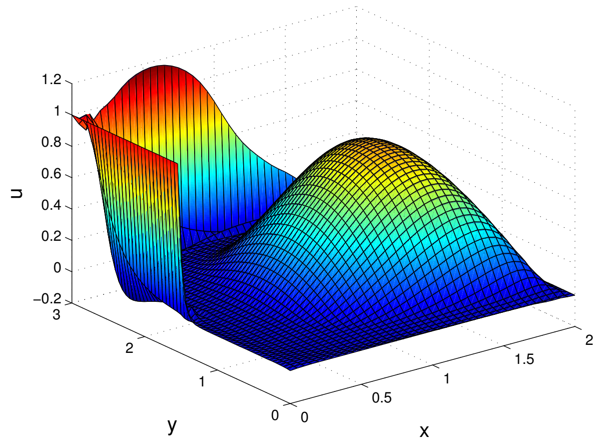


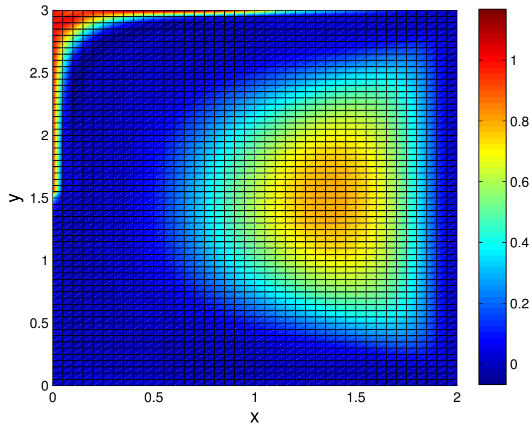
Figure 1.2: Convergence analysis of the iterative scheme using a tolerance of 10^{-4} and $h = 0.1$.

The final converged results of the Stokes and Transport problem can be seem in Figures 1.3 and 1.4. It can be observed that the density results are highly affected by the velocity field. The density concentrates in the the same region as the big vortex present in the cavity problem, meaning that the density is being "carried" by the flow, which is an expected result. The velocity and pressure fields are not altered so much by coupling with the density equation. This happens because in the provided viscosity-density relation, the values of ρ do not affect the initial viscosity ν_0 enough so more visible results can be seem. In fact, considering the maximum values reached by density, the viscosity is only changed by a factor of 1.8.

It is important to note that this implementation is very sensible to the input parameters choosing. Some values of initial viscosity and diffusivity may not reach convergence, which also happens with more complicated relations between viscosity and density.

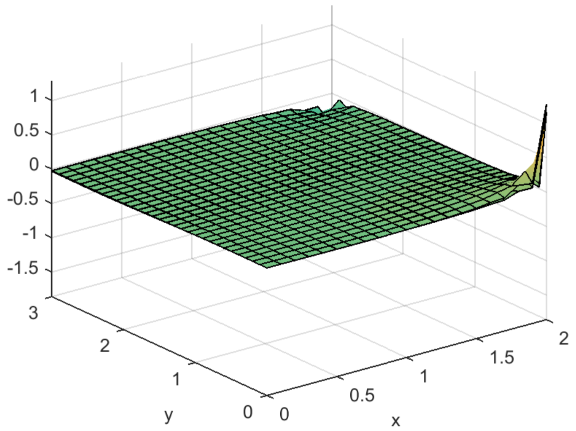


(a) Diagonal view

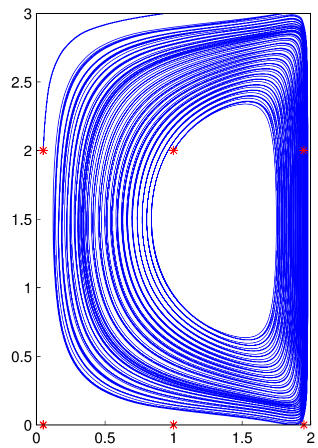


(b) Top view

Figure 1.3: Results for the density field of the coupled stokes-transport problem.



(a) Pressure field



(b) Velocity Streamlines

Figure 1.4: Results for the stokes equation field of the coupled problem.

2 Unsteady Coupling

For this case, a fully coupled approach was intended, although it couldn't be finished due to lack of time. Nevertheless, the linearisation and the conceptual approach followed is shown in this section.

2.1 Weak form and residual for Stokes equation

The strong form of the equation reads

$$u_t - \nabla \cdot (\nu \nabla) u - \nabla p = 0$$

Discretizing in time using a Backward Euler scheme the equation can be written in its residual form as

$$R_u = \frac{u^{n+1} - u^n}{\Delta t} - \nabla \cdot (\nu \nabla) u^{n+1} - \nabla p^{n+1} = 0$$

Integrating and weighting each member by a vectorial test function w

$$R_u = \int_\Omega w \cdot \frac{u^{n+1} - u^n}{\Delta t} d\Omega - \int_\Omega w \cdot \nabla \cdot (\nu \nabla) u^{n+1} d\Omega - \int_\Omega w \cdot \nabla p^{n+1} d\Omega = 0$$

Integrating by parts, and dropping the integration along the Neumann boundary (as all of them are of Dirichlet type)

$$R_u = \int_\Omega w \cdot \frac{u^{n+1} - u^n}{\Delta t} d\Omega + \int_\Omega \nu \nabla w : \nabla u^{n+1} d\Omega + \int_\Omega (\nabla \cdot w) p^{n+1} d\Omega = 0$$

We discretize in space with Galerkin method

$$u = u^h = N^u_{iB} u_{iB} \qquad p = p^h = N^P_B P_B \qquad w = N_{iA}$$

The final weak residual is

$$Ru = \int_\Omega N^u_{iA} \frac{N^u_{iB} u^{n+1}_{iB} - N^u_{iB} u^{n+1}_{iB}}{\Delta t} d\Omega + \int_\Omega \nu^h \nabla N^u_{iA} : \nabla N^u_{iB} u_{iB} d\Omega + \int_\Omega (\nabla \cdot N^u_{iA}) N^p_{iB} u_{iB} P_B d\Omega = 0$$

With the discretized viscosity law

$$\nu^h = \nu_0 + \nu_0 \frac{1}{(1 - exp(-10 N^\rho_C \rho_C - 0.5)}$$

2.2 Weak form and residual for Convection-Diffusion equation

The strong form of the equation reads

$$\rho_t - u \cdot \nabla \rho - \nabla \cdot (\mu \nabla \rho) = s(u)$$

Using the same time discretization, integrating and weighting (with a scalar test function) we obtain the following weak residual form

$$R_\rho = \int_\Omega v \frac{\rho^{n+1} - \rho^n}{\Delta t} d\Omega - \int_\Omega v \nabla \cdot (\mu \nabla) \rho^{n+1} d\Omega + \int_\Omega v u \cdot \nabla \rho^{n+1} d\Omega - \int_\Omega v s^{n+1}(u) = 0$$

The unknown and the weight are discretized as following

$$\rho = \rho^h = N^\rho_B \rho_B \qquad v = N^\rho_A$$

The final residual form is

$$R_\rho = \int_\Omega N^\rho_A \frac{N^\rho_B \rho^{n+1}_B - N^\rho_B \rho^n_B}{\Delta t} d\Omega + \int_\Omega \mu \nabla N^\rho_A \cdot \nabla N^\rho_B \rho^{n+1}_B + \int_\Omega N^\rho_A N^u_{iC} u^{n+1}_{iC} \nabla N^\rho_B \rho^{n+1}_B d\Omega \\ - \int_\Omega N^\rho_A \frac{1}{1 + exp(-10(||N^u_{iC} u^{n+1}_{iC}||) - 0.5)}$$

2.3 Linearisation

We derive each residual by the three variables, to get the following expressions

$$\frac{\partial R_u}{\partial u_{iB}} = \int_\Omega \frac{N^u_{iA} \cdot N^u_{iB}}{\Delta t} d\Omega + \int_\Omega \nu^h(\rho) \nabla N^u_{iA} : N^u_{iB} d\Omega$$

$$\frac{\partial R_u}{\partial P_B} = \int_\Omega (\nabla \cdot N^u_{iA}) N^p_B d\Omega$$

$$\frac{\partial R_u}{\partial \rho_C} = \int_\Omega \left[\frac{\nu_0 N^\rho_C 10 exp(N^\rho_C \rho^{n+1}_C - 0.5)}{exp(10(N^\rho_C \rho^{n+1}_C - 0.5)) + 1} \right] \nabla N^u_{iA} : \nabla N^u_{iB} u^{n+1}_{iB} d\Omega$$

$$\frac{\partial R_\rho}{\partial u_{iC}} = \int_\Omega N^\rho_A N^u_{iC} \nabla N^\rho_B \rho^{n+1}_B d\Omega - \int_\Omega N^\rho_A \left[\frac{10(N^u_{iC})^2 u^{n+1}_{iC} exp(10(||N^u_i c u^{n+1}_{iC}|| - 0.5))}{||N^u_{ic} u^{n+1}_{iC}||(exp(10(||N^u_{ic} u^{n+1}_{iC}|| - 0.5) + 1)^2)} \right] d\Omega$$

$$\frac{dR_\rho}{dp_B} = 0$$

$$\frac{\partial R_\rho}{\partial \rho_B} = \int_\Omega N^\rho_A N^\rho_B \frac{1}{\Delta t} d\Omega + \int_\Omega \mu \nabla N^\rho_A \cdot \nabla N^\rho_B d\Omega + \int_\Omega N^\rho_A N^u_{iC} u^{n+1}_{iC} \nabla N^\rho_B d\Omega$$

We can assemble all the differentiated residuals in a partitioned matrix, to obtain the following system of equations.

$$\begin{bmatrix} \frac{\partial R_u}{\partial u} & \frac{\partial R_u}{\partial p} & \frac{\partial R_u}{\partial \rho} \\ \frac{\partial R_\rho}{\partial u} & \frac{\partial R_\rho}{\partial p} & \frac{\partial R_\rho}{\partial \rho} \end{bmatrix} \begin{bmatrix} \delta u \\ \delta p \\ \delta \rho \end{bmatrix} = \begin{bmatrix} R_u \\ R_\rho \end{bmatrix}$$

Imposing the residuals to be zero, the system is solved and the variables are updated in each iteration as following.

$$\begin{aligned} u^{n+1}_{k+1} &= u^{n+1}_k + \delta u \\ p^{n+1}_{k+1} &= p^{n+1}_k + \delta p \\ \rho^{n+1}_{k+1} &= \rho^{n+1}_k + \delta \rho \end{aligned}$$