2D-Unsteady Transport Problem

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I. The behavior of the methods

1. Question 1

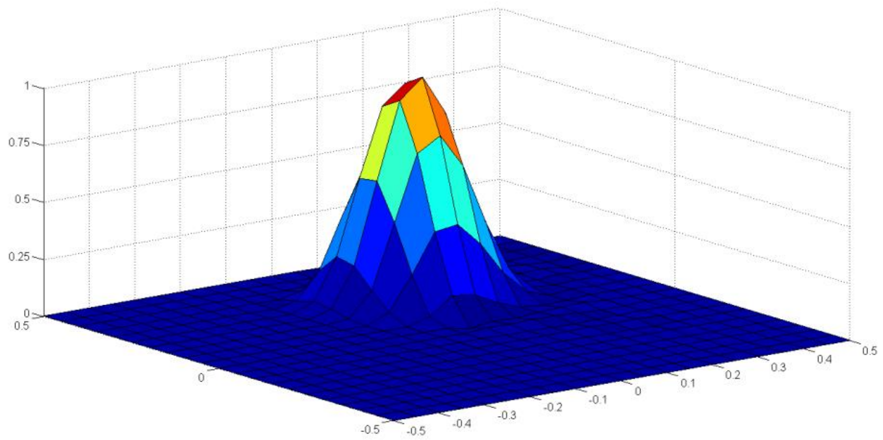

Figure 1. The result of Lax-Wendroff + Galerkin method

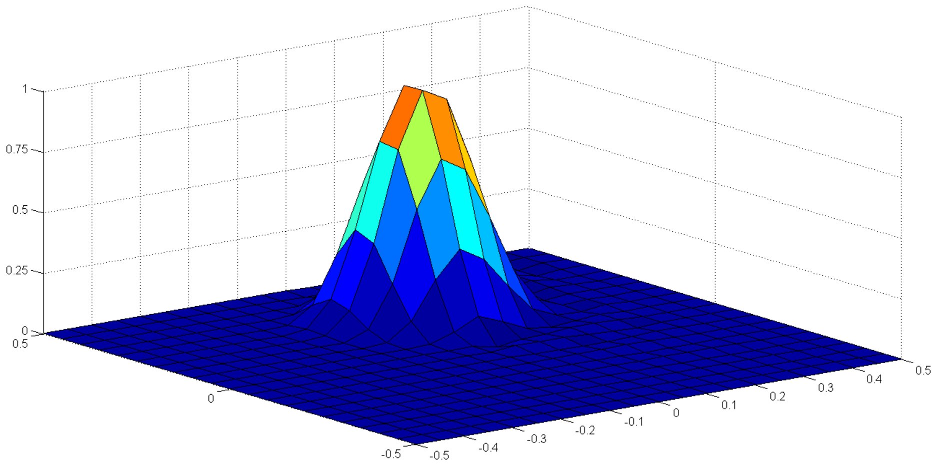

Figure 2. The result of TG3 method

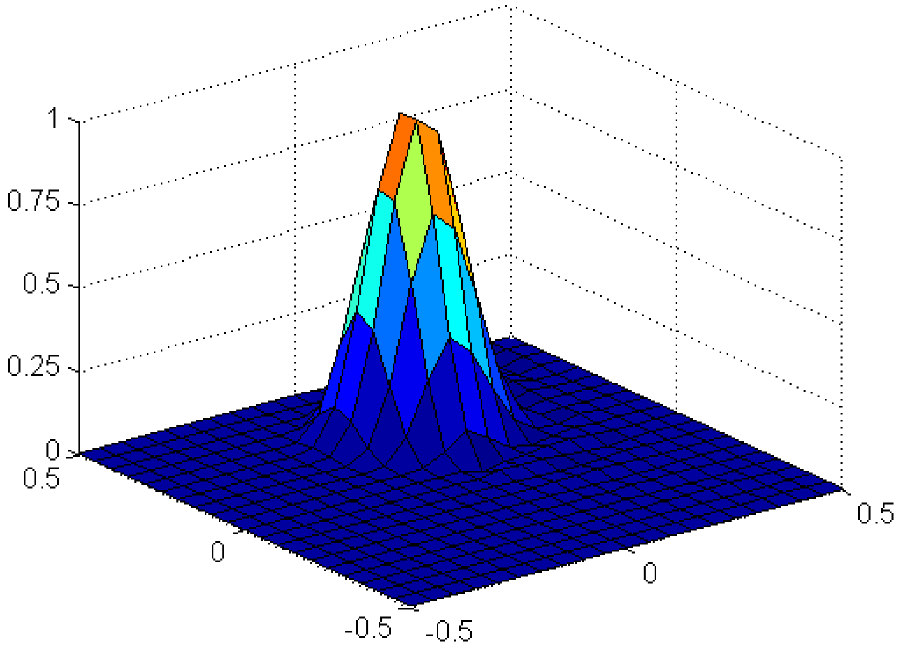

Figure 3. The result of CN method

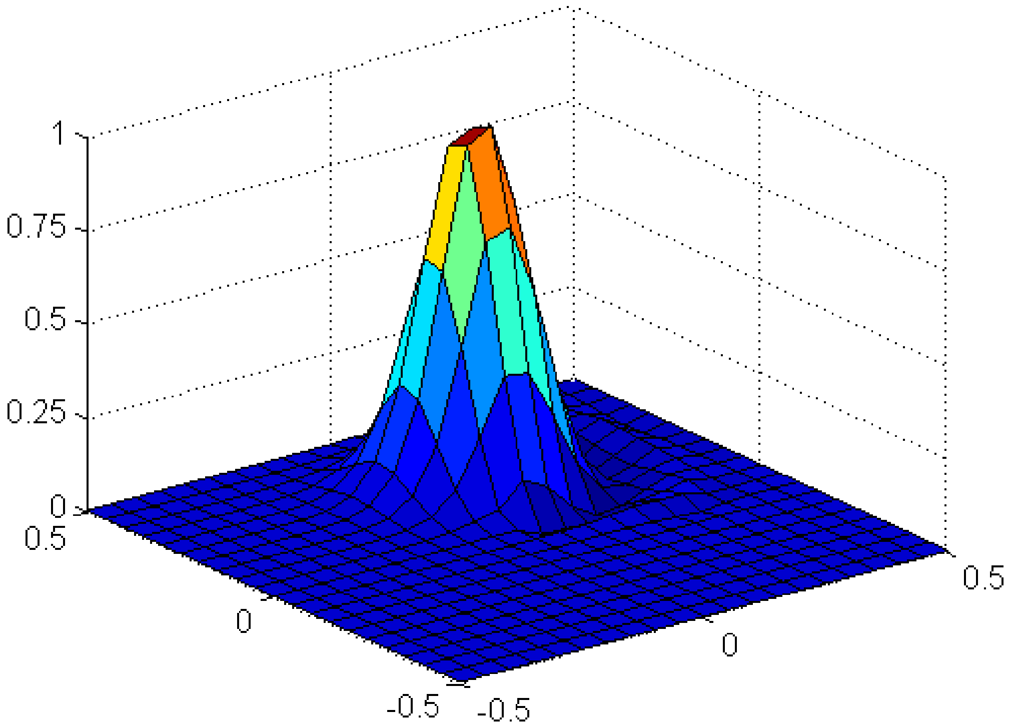

Figure 4. The result of CN-FD method

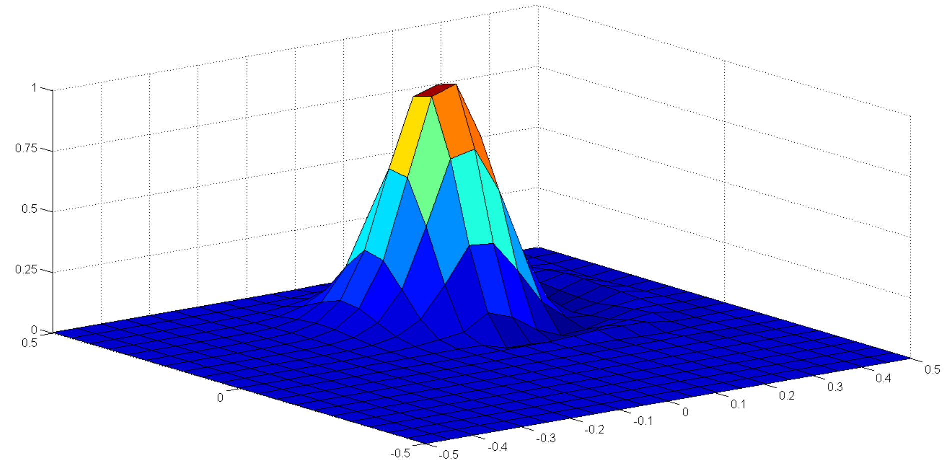

Figure 5. The result of LW-FD method

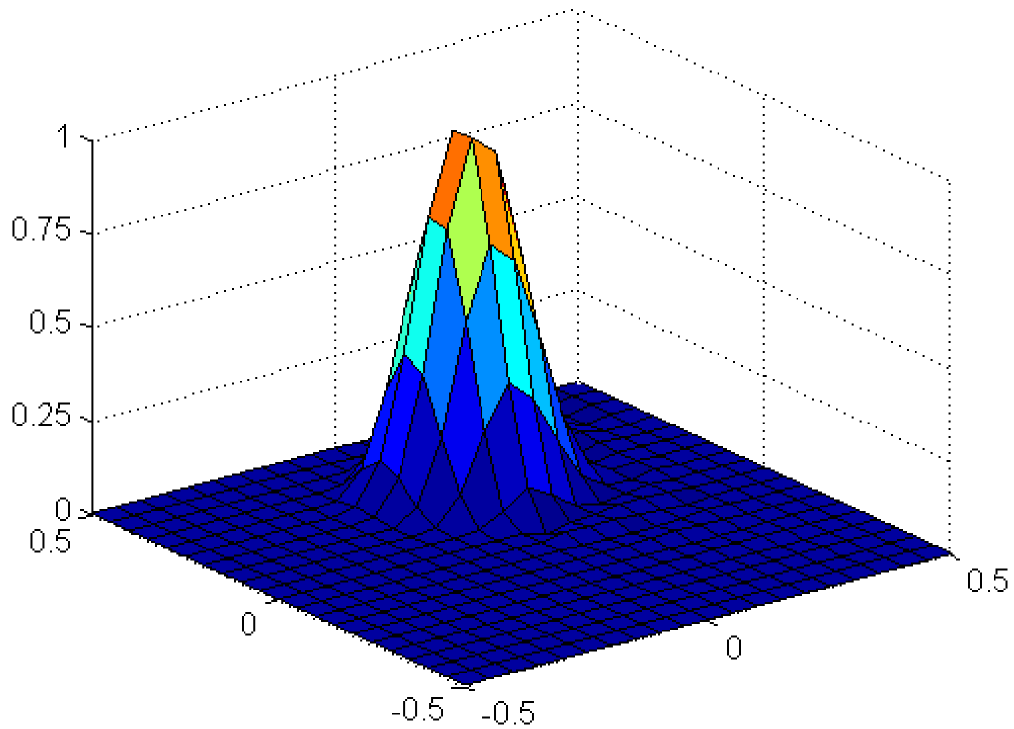

Figure 6. The result of TG4 method

For question 1, the velocity field is $v(x, y) = (-y, x)$. Under this velocity field, all the methods behave well. But the LW-FD method and CN-FD method have rather poor behavior than other methods because they show more oscillations and we will find this problem in question 2.

2. Question 2

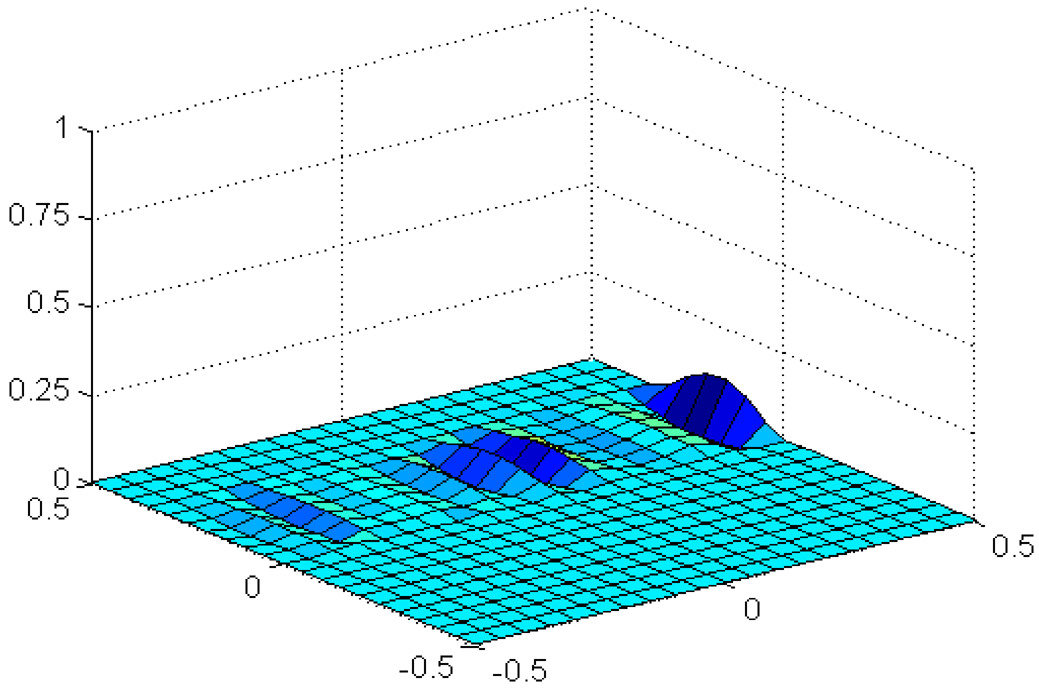

Figure 7. The result of CN method of question 2

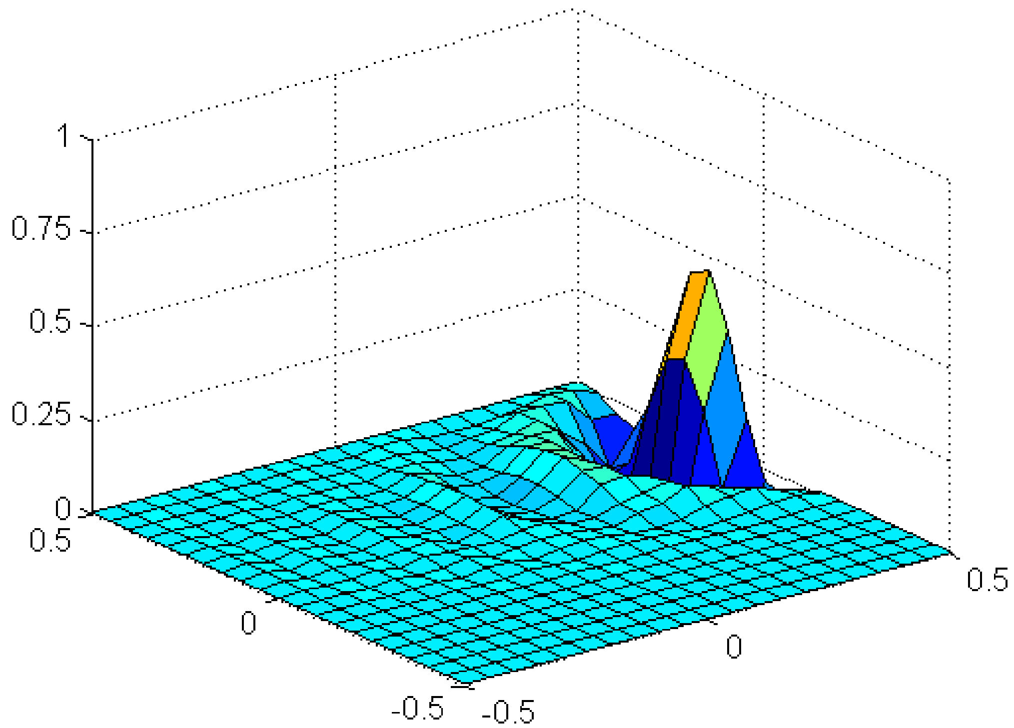

Figure 8. The result of CN-FD method of question 2

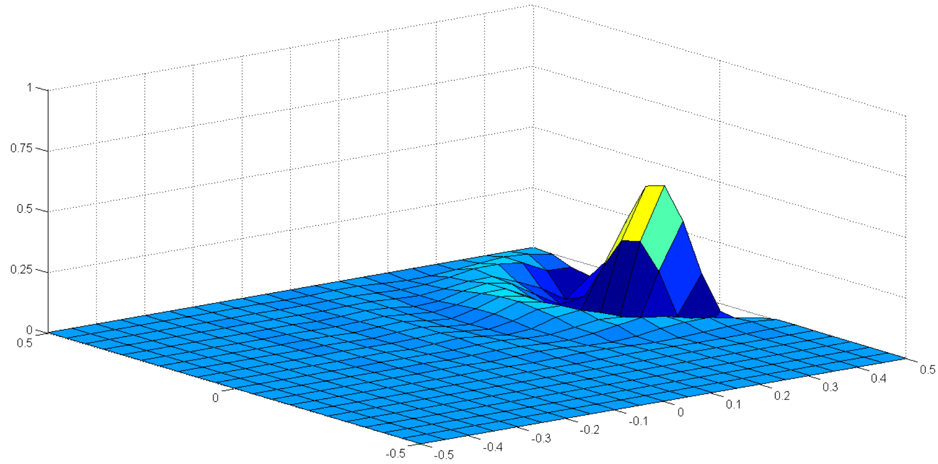

Figure 9. The result of LW-FD method of question 2

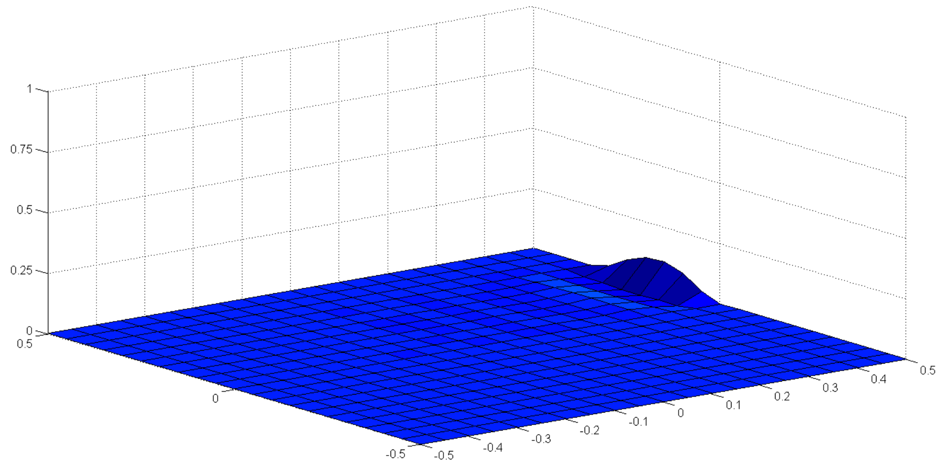

Figure 10. The result of TG2 method of question 2

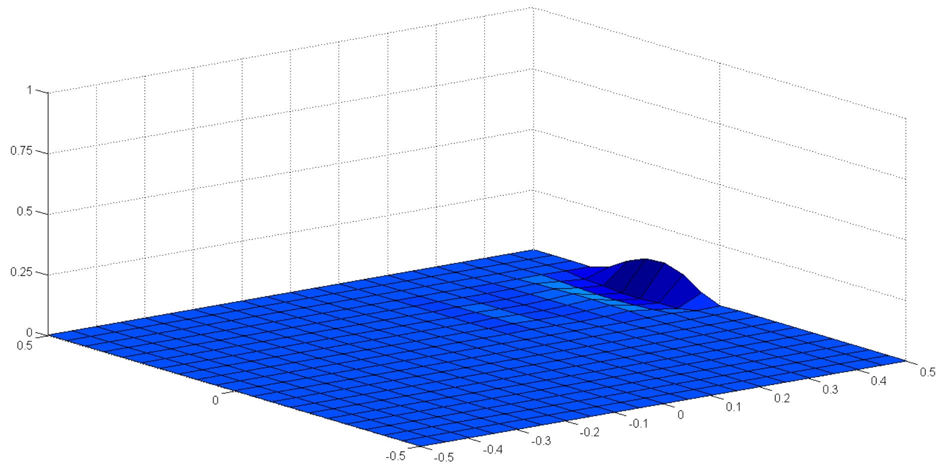

Figure 11. The result of TG3 method of question 2

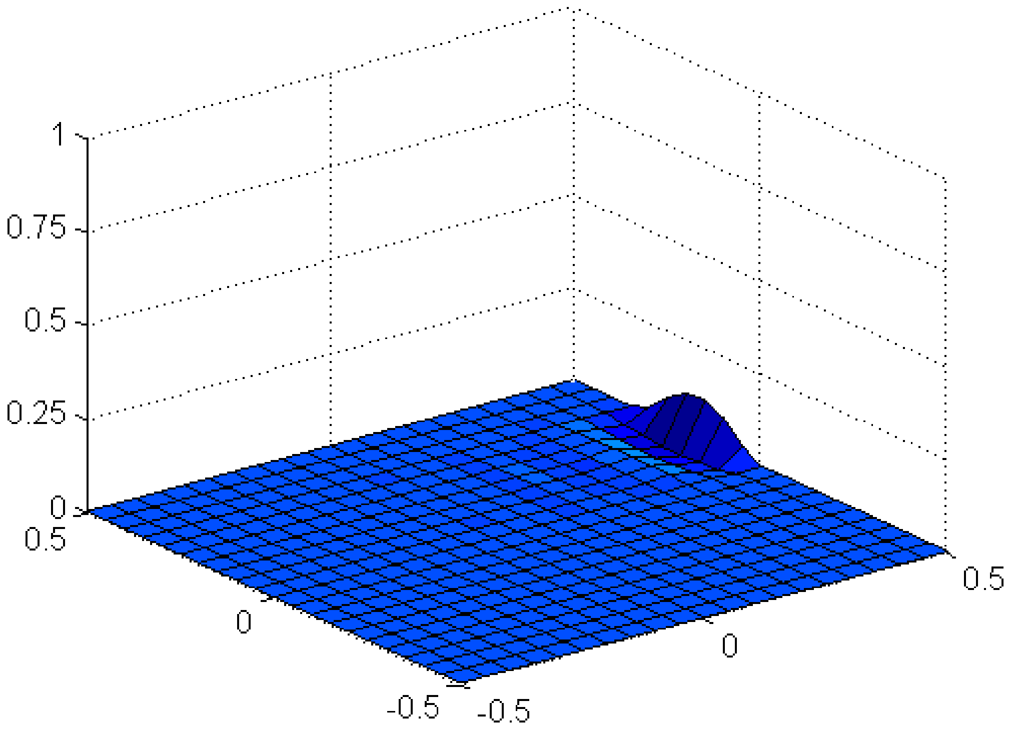

Figure 12. The result of TG-2S method of question 2

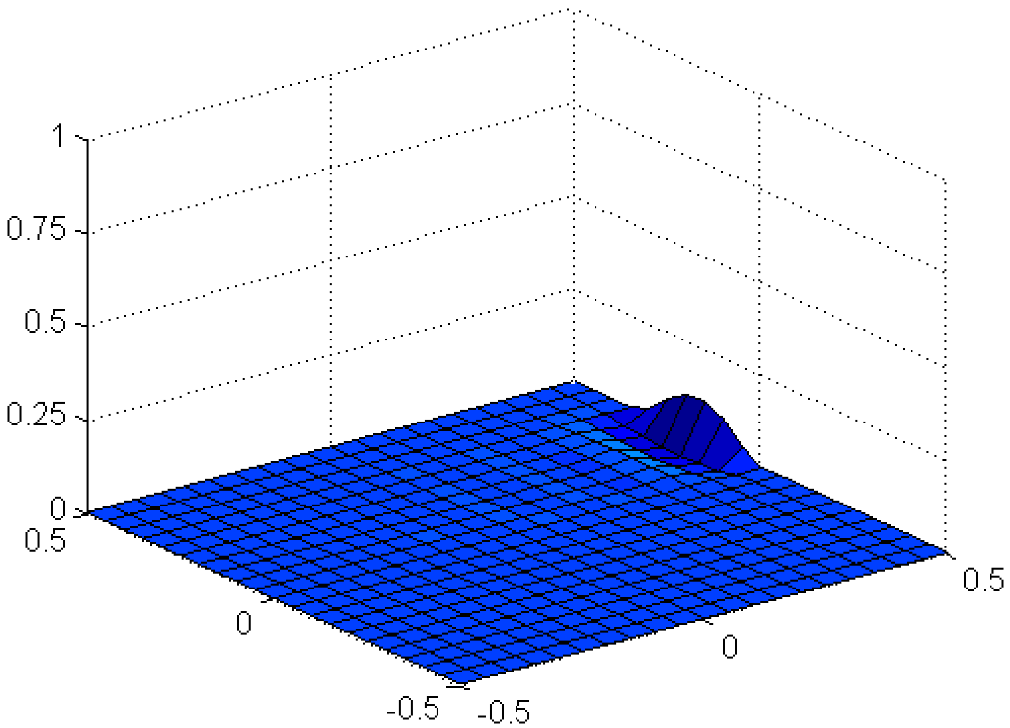

Figure 13. The result of TG4-2S method of question 2

For question 2, the velocity is $v(x,y) = (1,0)$. In above results, we can find that the TG2, TG3, TG3-2S, TG4-2S have expected behaviors. But the CN method shows oscillations which means that it's unstable for question 2. For CN-FD and LW-FD, they both work badly. Their results not only oscillate in the domain, but also lose accuracies compared to other methods. For CN and CN-FD method, the reason why they work badly is that they have more phase errors. And for LW-FD, it's because LW-FD uses the diagonal stiffness matrix so it will lose information, which make the method inaccurate.

3. Question 3

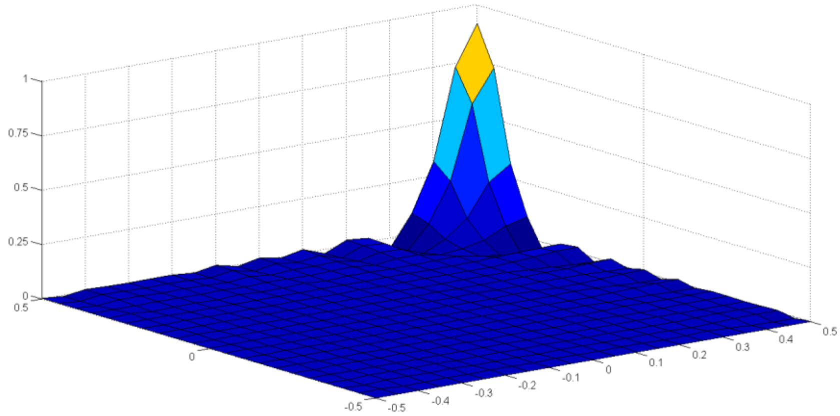

Figure 14. The result of CN method of question 3

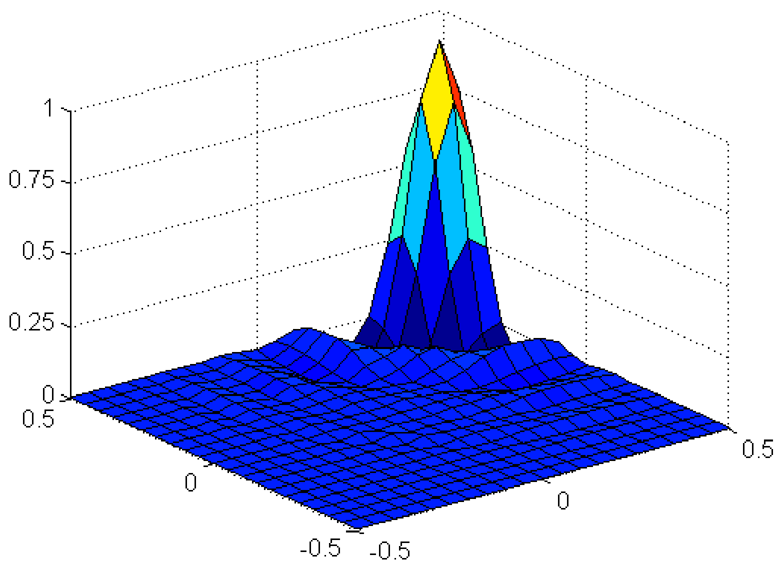

Figure 15. The result of CN-FD method of question 3

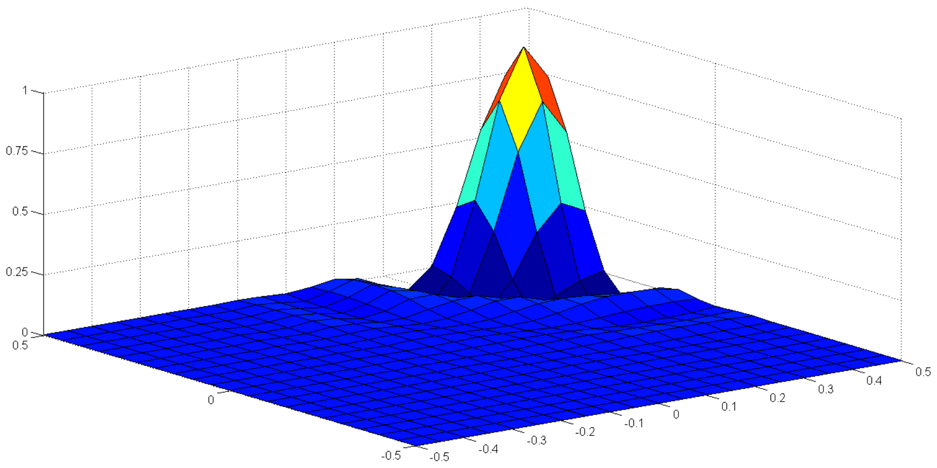

Figure 16. The result of LW-FD method of question 3

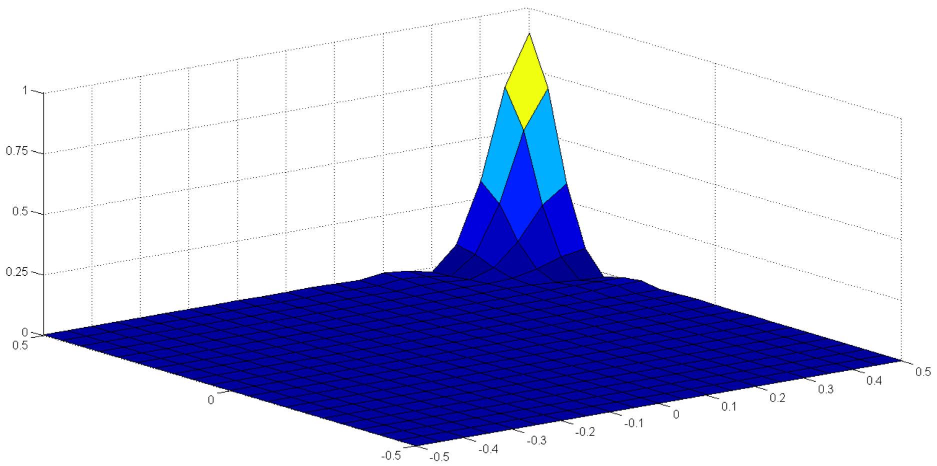

Figure 17. The result of TG2 method of question 3

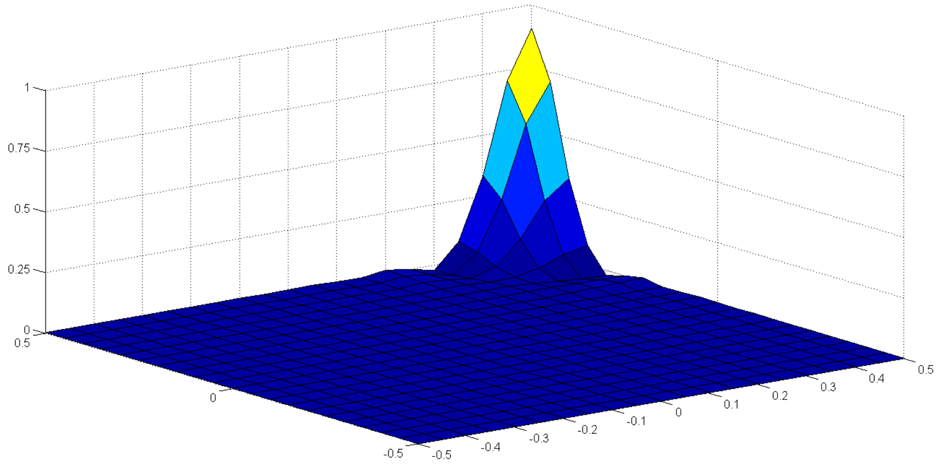

Figure 18. The result of TG3 method of question 3

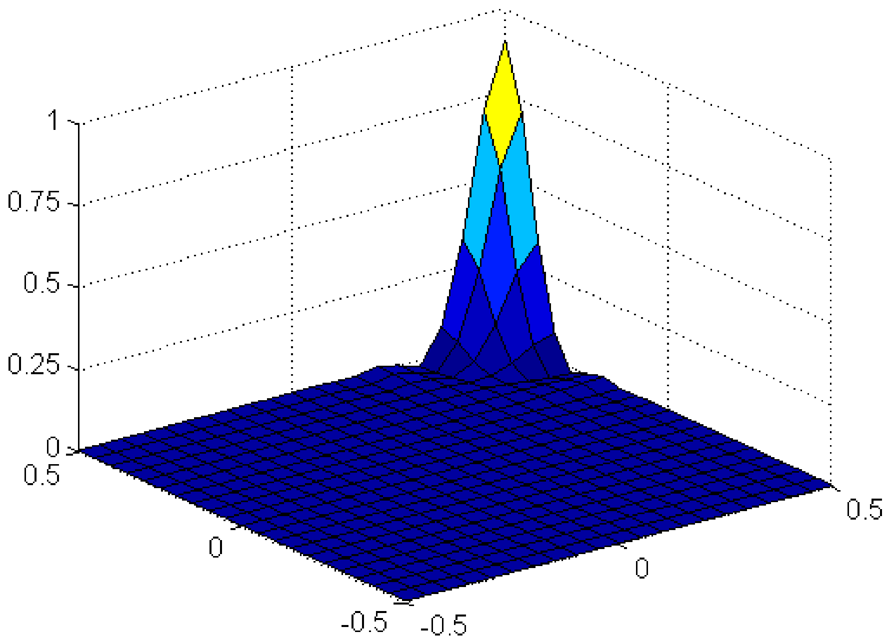

Figure 19. The result of TG3-2S method of question 3

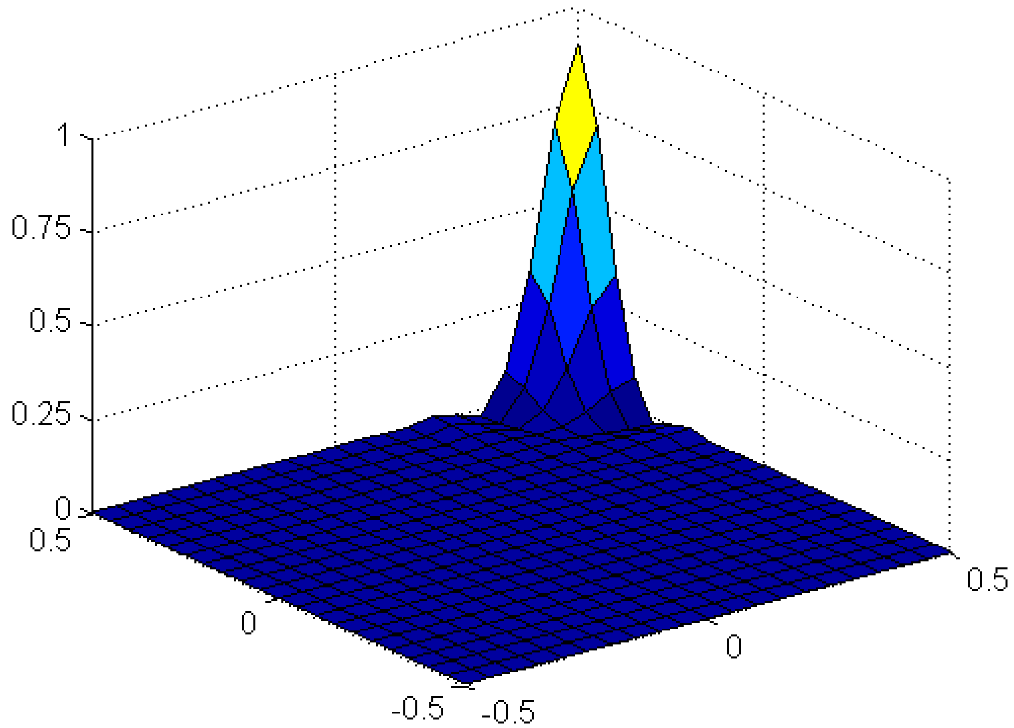

Figure 20. The result of TG4-2S method of question 3

For question 3, the velocity field is $v(x,y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Like question 2, the TG2, TG3, TG3-2S, TG4-2S work better than CN, CN-FD and LW-FD and the reason is same to question 2. The result of CN shows oscillation and the results of CN-FD and LW-FD shows both oscillations and inaccuracies.

II. The comments on the code

Because the source term is zero in this part, so we ignore the f in following methods. And the code representations are like following:

$$\boldsymbol{K}\ represents\ the\ term:\ (\boldsymbol{a}\cdot\nabla N_i\ ,\boldsymbol{a}\cdot\nabla N_j)$$

$$\boldsymbol{C}\ represents\ the\ term:(N_i\ ,\boldsymbol{a}\cdot\nabla N_j)$$

$$\boldsymbol{M}\ represents\ the\ term:(N_i\ ,N_j)$$

$$\boldsymbol{M_O}\ represents\ the\ term\ :\left((\boldsymbol{a}\cdot\boldsymbol{n})N_i\ ,N_j\right)\ \ on\ \ \Gamma_{out}$$

$\boldsymbol{C_O}$ *represents the term* : $\left((\boldsymbol{a}\cdot\boldsymbol{n})N_i\,,\boldsymbol{a}\cdot\nabla N_j\right)$ *on* Γ_{out}

$\boldsymbol{M_d}$ *represents diagonal mass matrix of* $\boldsymbol{M}$

$\boldsymbol{M_{od}}$ *represents the diagonal mass matrix of* $\boldsymbol{M_O}$

1. LW method and LW-FD method

The time and spatial discretization:

$$\left(\omega,\frac{\Delta u}{\Delta t}\right)=\left(\boldsymbol{a}\cdot\boldsymbol{\nabla}\omega,u^n-\frac{\Delta t}{2}\cdot(\boldsymbol{a}\cdot\boldsymbol{\nabla})u^n\right)+\left((\boldsymbol{a}\cdot\boldsymbol{n})\omega,u^n+\frac{\Delta t}{2}\cdot\boldsymbol{\nabla}u^n\right)_{\Gamma_{out}}$$

The code is：

LW method:

```
A = M;
B = dt*(C - (dt/2)*K - Mo + (dt/2)*Co);
f = dt*(v1 + (dt/2)*(v2-vo));
```

LW-FD mehod:

```
Md = diag(M*ones(numnp, 1));
Mod = diag(Mo*ones(numnp, 1));
A = Md;
B = dt*(C - (dt/2)*K - Mod + (dt/2)*Co);
f = dt*(v1 + (dt/2)*(v2-vo));
```

2. CN method and CN-FD method

The time and spatial discretization:

$$\left(\omega,\frac{\Delta u}{\Delta t}\right)-\left(\mathbf{a}\cdot\boldsymbol{\nabla}\omega,\frac{1}{2}\cdot\Delta u\right)+\left((\boldsymbol{a}\cdot\boldsymbol{n})\omega,\frac{1}{2}\cdot\Delta u\right)_{\Gamma_{out}}$$

$$=(\boldsymbol{a}\cdot\boldsymbol{\nabla}\omega,u^n)+\left((\boldsymbol{a}\cdot\boldsymbol{n})\omega,u^n\right)_{\Gamma_{out}}$$

The code is:

CN method:

```
A = M - (dt/2)*C + (dt/2)*Mo;
B = dt*C - dt*Mo;
f = dt*v1;
```

CN-FD method:

```
Md = diag(M*ones(numnp, 1));
Mod = diag(Mo*ones(numnp, 1));
A = Md - (dt/2)*C + (dt/2)*Mod;
B = dt*C - dt*Mod;
f = dt*v1;
```

3. TG3 method

The time and spatial discretization:
We impose:

$$g = \frac{u^{n+1} - u^n}{\Delta t}$$

We get:

$$(\omega, g) + \left(\boldsymbol{a} \cdot \boldsymbol{\nabla}\omega, \frac{\Delta t^2}{6} \cdot (\boldsymbol{a} \cdot \boldsymbol{\nabla})g\right) - \left((\boldsymbol{a} \cdot \boldsymbol{n})\omega, \frac{\Delta t^2}{6} \cdot (\boldsymbol{a} \cdot \boldsymbol{\nabla})g\right)_{\Gamma_{out}} = \left(\boldsymbol{a} \cdot \boldsymbol{\nabla}\omega, u^n - \frac{\Delta t}{2} \cdot (\boldsymbol{a} \cdot \boldsymbol{\nabla})u^n\right) + \left((\boldsymbol{a} \cdot \boldsymbol{n})\omega, u^n + \frac{\Delta t}{2} \cdot (\boldsymbol{a} \cdot \boldsymbol{\nabla})u^n\right)_{\Gamma_{out}}$$

The code is:

```
A = M +(dt^2/6)*(K - Co);
B = dt*(C - (dt/2)*K - Mo + (dt/2)*Co);
f = dt*((dt/2)*(v2 - vo) +v1);
```

4. TG3-2S method

The time and spatial discretization:

Step 1:

$$(\omega, \tilde{u}^n) = (\omega, u^n) + \left(\mathbf{a} \cdot \boldsymbol{\nabla}\omega, \frac{\Delta t}{3} \cdot u^n - \alpha \cdot \Delta t^2 \cdot (\boldsymbol{a} \cdot \boldsymbol{\nabla})u^n\right) - \left((\boldsymbol{a} \cdot \boldsymbol{n})\omega, \frac{\Delta t}{3} \cdot u^n - \alpha \cdot \Delta t^2 \cdot (\boldsymbol{a} \cdot \boldsymbol{\nabla})u^n\right)_{\Gamma_{out}}$$

Where $\alpha = \frac{1}{9}$

The code is:

```
A1 = M;
B1 = -(dt/3)*C'- alpha*dt^2*(K - Co);
f1 = (dt/3)*v1 + alpha*dt^2*(v2 - vo);
```

Step 2:

$$(\omega, \Delta u) = \left(\mathbf{a} \cdot \boldsymbol{\nabla}\omega, \Delta t \cdot u^n - \frac{1}{2} \cdot \Delta t^2 \cdot (\boldsymbol{a} \cdot \boldsymbol{\nabla}) u^n\right) - \left((\boldsymbol{a} \cdot \boldsymbol{n})\omega, \Delta t \cdot u^n - \frac{1}{2} \cdot \Delta t^2 \cdot (\boldsymbol{a} \cdot \boldsymbol{\nabla}) u^n\right)_{\Gamma_{out}}$$

The code is:

```
A2 = M;
B2 = -dt*C';
C2 = - (dt^2/2)*(K-Co);
f2 = dt*v1 - (dt^2/2)*(v2 - vo);
```

5. TG4-2S

The TG4-2S method is same to TG3-2S method except that the α in TG4-2S method is equal to $\frac{1}{12}$.