B. 2D Unsteady Transport

Define Problem:

2D homogeneous convection equation with initial condition and homogeneous Dirichlet conditions on the inlet boundary

$U(x, 0) = 0.25 * (1 + \cos \pi X_1)(1 + \cos \pi X_2)$ If $X_1^{\,2} + X_2^{\,2} \leq 1$

$U(x, 0) = 0$

Introduction:

The numerical solution has been computed using following finite element schemes:

i. Lax-Wendroff + Galerkin (and with lumped mass matrix);
ii. Crank-Nicolson +Galerkin (and lumped mass matrix);
iii. The third-order explicit Taylor-Galerkin scheme (TG3);
iv. Two-step third order Taylor-Galerkin-2S method (TG3-2S);
v. Two-step fourth order method (TG4-2S).

>> Term Represented in the supplied code:

a) Lax-Wendroff Method:

$$\frac{\Delta u}{\Delta t} = -\mathbf{a}.\boldsymbol{\nabla} u^n + \frac{\Delta t}{2}(\mathbf{a}.\boldsymbol{\nabla})^2 u^n + s^n + \frac{\Delta t}{2}({s_t}^n - \mathbf{a}.\boldsymbol{\nabla} s^n)$$

The weak form for the Lax-Wendroff method can be written as considering $s = 0$ and $h = 0$:

$$(w, \frac{\Delta u}{\Delta t}) = (\mathbf{a}.\boldsymbol{\nabla} w, u^n - \frac{\Delta t}{2}(\mathbf{a}.\boldsymbol{\nabla})u^n) - ((\mathbf{a}.\mathbf{n})w, u^n - \frac{\Delta t}{2}(\mathbf{a}.\boldsymbol{\nabla})u^n)_{\Gamma^{out}}$$

In the supplied code : A = M represents the term $(w, \Delta u)$;

In, B $= dt * (C - (dt/2) * K - Mo + (dt/2) * Co); C\ represents\ the\ term\ (\mathbf{a}.\boldsymbol{\nabla} w, u^n)$

$K\ represents\ the\ term\ (\mathbf{a}.\boldsymbol{\nabla} w)(\mathbf{a}.\boldsymbol{\nabla} u^n)$

$Mo\ represents\ ((\mathbf{a}.\mathbf{n})w, u^n)_{\Gamma^{out}}$

Co represents $((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\boldsymbol{\nabla})u^n)_{\Gamma^{out}}$

b. Crank Nicolson Method:

$\frac{\Delta u}{\Delta t} + \frac{1}{2}(\mathbf{a}.\mathbf{\nabla})\Delta u = -\mathbf{a}.\mathbf{\nabla} u^n$

The weak form for the CN method can be written as considering $s = 0$ and $h = 0$:

$(w, \frac{\Delta u}{\Delta t}) - \frac{1}{2}(\mathbf{\nabla} w, \mathbf{a}\Delta u) + \frac{1}{2}((\mathbf{a}.\mathbf{n})w, \Delta u)_{\Gamma^{out}} = (\mathbf{\nabla} w, \mathbf{a}u^n) - ((\mathbf{a}.\mathbf{n})w, u^n)_{\Gamma^{out}}$

In the supplied code for the Galerkin formulation + Crank-Nicolson with consistent mass matrix:

In $A = M-(dt/2)*C+(dt/2)*Mo$: M represents the term $(w, \Delta u)$,C *represents the term* $(\mathbf{\nabla} w, \mathbf{a}\Delta u)$and Mo represents $((\mathbf{a}.\mathbf{n})w, \Delta u)_{\Gamma^{out}}$;

In, B $= dt * C - dt * Mo : C$ *represents the term* $(\mathbf{a}.\mathbf{\nabla} w, u^n)$

Mo *represents* $((\mathbf{a}.\mathbf{n})w, u^n)_{\Gamma^{out}}$

Co represents $((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\mathbf{\nabla})u^n)_{\Gamma^{out}}$

As s=0; it is not required to describe the terms in $f = dt * v1$;

c. Third Order Taylor Galerkin Method (TG3):

$(1 - \frac{\Delta t^2}{6}(\mathbf{a}.\mathbf{\nabla})^2)\frac{\Delta u}{\Delta t} = -\mathbf{a}.\mathbf{\nabla} u^n + \frac{\Delta t}{2}(\mathbf{a}.\mathbf{\nabla})^2 u^n$

The weak form for the TG3 method can be written as considering $s = 0$ and $h = 0$:

$(w, \frac{\Delta u}{\Delta t}) + \frac{\Delta t^2}{6}(\mathbf{a}.\mathbf{\nabla} w, (\mathbf{a}.\mathbf{\nabla}\frac{\Delta u}{\Delta t}) - \frac{\Delta t^2}{6}((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\mathbf{\nabla}\frac{\Delta u}{\Delta t}))_{\Gamma^{out}}$

$= (\mathbf{a}.\mathbf{\nabla} w, u^n - \frac{\Delta t}{2}(\mathbf{a}.\mathbf{\nabla})u^n) - ((\mathbf{a}.\mathbf{n})w, u^n - \frac{\Delta t}{2}(\mathbf{a}.\mathbf{\nabla})u^n)_{\Gamma^{out}}$

In the supplied code :

In $A = M+(dt^2/6)*(K-Co)$: M represents the term $(w, \Delta u)$,K *represents the term* $(\mathbf{a}.\mathbf{\nabla} w)(\mathbf{a}.\mathbf{\nabla}\Delta u)$and Co represents $((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\mathbf{\nabla})\Delta u)_{\Gamma^{out}}$;

In, B $= dt * (C - (dt/2) * K - Mo + (dt/2) * Co) : C$ *represents the term* $(\mathbf{a}.\mathbf{\nabla} w, u^n)$

K *represents the term* $(\mathbf{a}.\mathbf{\nabla} w)(\mathbf{a}.\mathbf{\nabla} u^n)$

Mo *represents* $((\mathbf{a}.\mathbf{n})w, u^n)_{\Gamma^{out}}$

Co represents $((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\mathbf{\nabla})u^n)_{\Gamma^{out}}$

As s=0; it is not required to describe the terms in $f = dt * ((dt/2) * (v2 - vo) + v1)$;

d. Two Step Third Order Taylor Galerkin Method (TG3-2S):

The weak form of the TG3-2S scheme can be expressed as with $\alpha = 1/9$:

$$(w, \frac{\bar{u}^n - u^n}{\Delta t}) = \frac{1}{3}(\mathbf{a}.\mathbf{\nabla} w, u^n) - \alpha(\Delta t)(\mathbf{a}.\mathbf{\nabla} w, \mathbf{a}.\mathbf{\nabla} u^n) + \alpha(\Delta t)((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\mathbf{\nabla})u^n)_{\Gamma^{out}}$$

$$(w, \frac{u^{n+1} - u^n}{\Delta t}) = (\mathbf{a}.\mathbf{\nabla} w, u^n) - \frac{\Delta t}{2}(\mathbf{a}.\mathbf{\nabla} w, \mathbf{a}.\mathbf{\nabla} \bar{u}^n) + \frac{\Delta t}{2}((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\mathbf{\nabla})\bar{u}^n)_{\Gamma^{out}}$$

In the supplied code for the TG3-2S:

In $A1 = M$: M represents the term $(w, \bar{u}^n - u^n)$;

In, B1 $= -(dt/3) * C' - alpha * dt^2 * (K - Co) : -C'$ *represents the term* $(\mathbf{a}.\mathbf{\nabla} w, u^n)$

K *represents* $(\mathbf{a}.\mathbf{\nabla} w, \mathbf{a}.\mathbf{\nabla} u^n)$

Co represents $((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\mathbf{\nabla})u^n)_{\Gamma^{out}}$

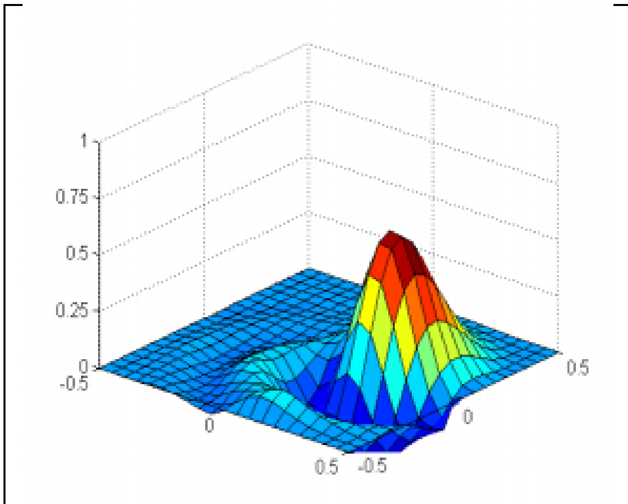
In $A2 = M$: M represents the term $(w, u^{n+1} - u^n)$;

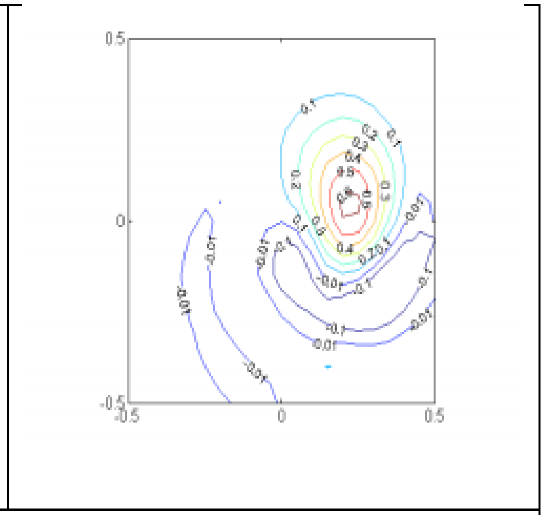
In, B2 $= -dt * C' : -C'$ *represents the term* $(\mathbf{a}.\mathbf{\nabla} w, u^n)$

In $C2 = -(dt^2/2) * (K - Co)$; K *represents* $(\mathbf{a}.\mathbf{\nabla} w, \mathbf{a}.\mathbf{\nabla} \bar{u}^n)$

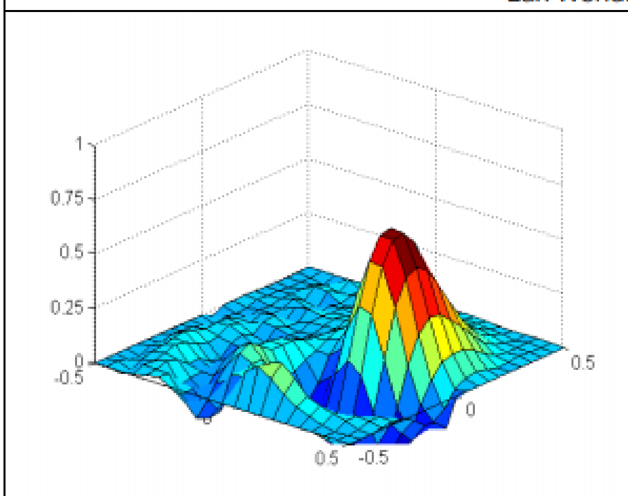
Co represents $((\mathbf{a}.\mathbf{n})w, (\mathbf{a}.\mathbf{\nabla})\bar{u}^n)_{\Gamma^{out}}$

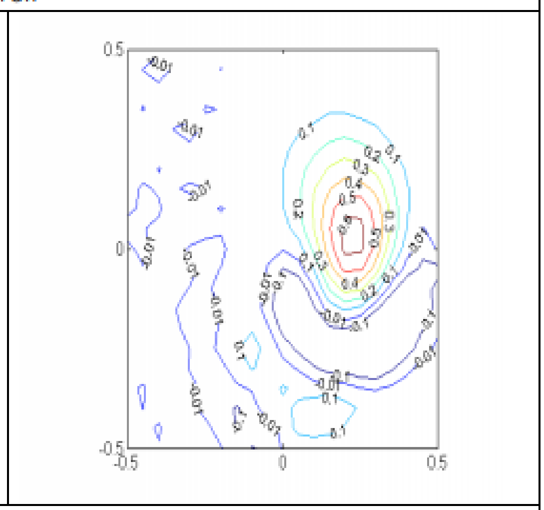
Graphs: Implication of finite element schemes , Convection of cosine hill in pure rotation velocity field.



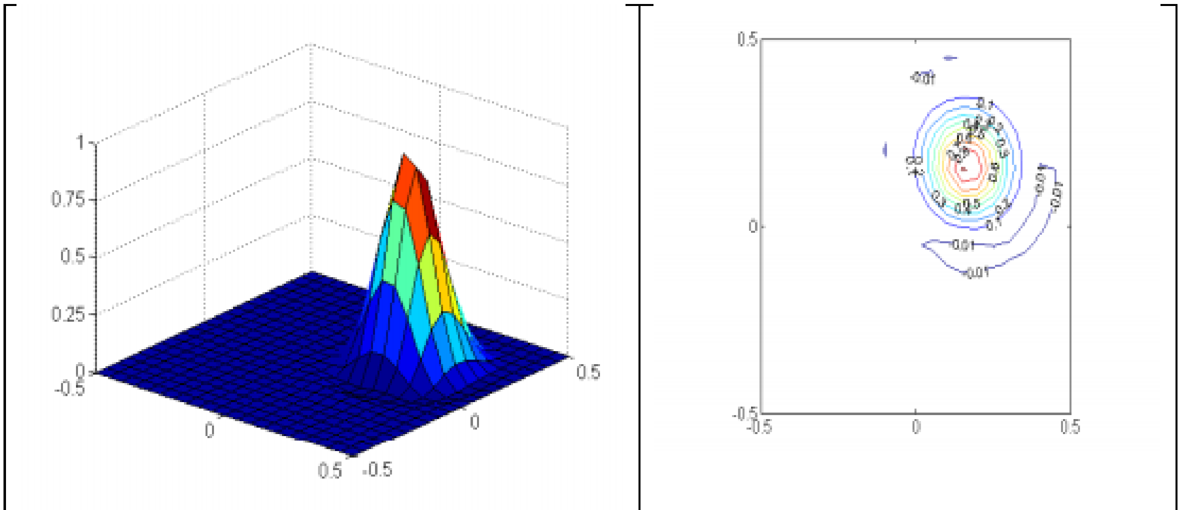


Lax Wendroff

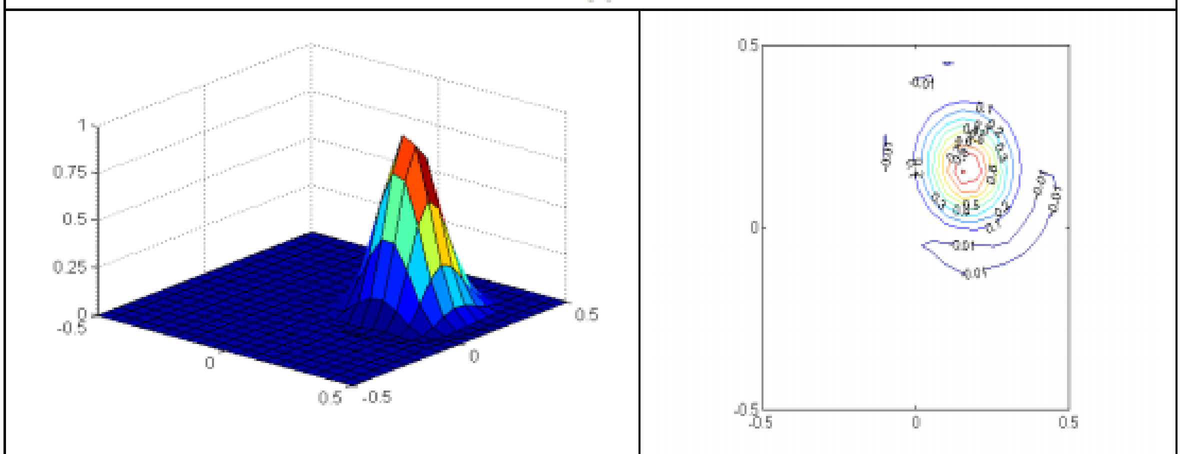




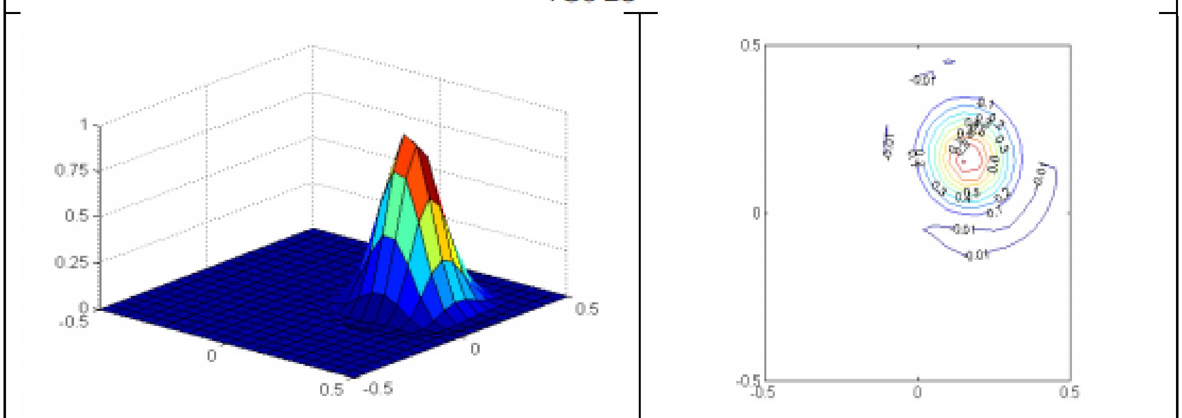
Crank Nicolson



TG3



TG3 2S



TG4 2S

Codes:

```
if meth == 1
A = M;
B = dt*(C - (dt/2)*K - Mo + (dt/2)*Co);
f = dt*(v1 + (dt/2)*(v2-vo));
elseif meth == 2
Md = diag(M*ones(numnp,1));
Mod = diag(Mo*ones(numnp,1));
A = Md;
B = dt*(C - (dt/2)*K - Mod + (dt/2)*Co);
f = dt*(v1 + (dt/2)*(v2-vo));
elseif meth == 3
A = M +(dt^2/6)*(K - Co);
B = dt*(C - (dt/2)*K - Mo + (dt/2)*Co);
f = dt*((dt/2)*(v2 - vo) +v1);
elseif meth == 4
A = M - (dt/2)*C + (dt/2)*Mo;
B = dt*C - dt*Mo;
f = dt*v1;
elseif meth == 5
Md = diag(M*ones(numnp,1));
Mod = diag(Mo*ones(numnp,1));
A = Md - (dt/2)*C + (dt/2)*Mod;
B = dt*C - dt*Mod;
f = dt*v1;
elseif meth == 6
A = M + (dt/2)*(C + C') + (dt^2/4)*K ;
B = -dt*(C' + (dt/2)*K);
f = dt*v1 + (dt^2/2)*v2;
elseif meth == 7 %TG3-2S
alpha = 1/9;
A1 = M;
B1 = -(dt/3)*C'- alpha*dt^2*(K - Co);
f1 = (dt/3)*v1 + alpha*dt^2*(v2 - vo);
A2 = M;
B2 = -dt*C';
C2 = - (dt^2/2)*(K-Co);
f2 = dt*v1 - (dt^2/2)*(v2 - vo);
elseif meth == 8 %TG4-2S
alpha = 1/12;
A1 = M;
B1 = -(dt/3)*C'- alpha*dt^2*(K - Co);
f1 = (dt/3)*v1 + alpha*dt^2*(v2 - vo);
A2 = M;
B2 = -dt*C';
C2 = - (dt^2/2)*(K-Co);
f2 = dt*v1 - (dt^2/2)*(v2 - vo);
else
error('Unavailable method')
end
```