2D Steady-Unsteady Transport
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2D Steady Unsteady Convention-Diffusion Problem.

A. 2D Steady Transport

Define Problem:

1. A convection-reaction dominated case with
   ||*a*||=1/2, ν=10-4, σ=1.
2. A reaction dominated case with
   ||*a*||=10-3, ν=10-4, σ=1.

Introduction: To solve the defined problem using formulation such as Galerkin, Streamline Upwind (SU), Streamline Upwind Petrov-Galerkin (SPUG) and Galerkin Least Square (GLS) Method.

Code: The Matlab code for Galerkin, Artificial Diffusion & SUPG was already given and some changes are made in the code to get run GLS method. The changes are as follow,

```
elseif method == 3
    % GLS
    Ke = Ke + (nu*(Nx'*Nx+Ny'*Ny) + N_ig'*(ax*Nx+ay*Ny) + N_ig'*sgma*N_ig ...
        + tau*(nu*(Nx+Ny) +(ax*Nx+ay*Ny))'*(ax*Nx+ay*Ny))*dvolu;
    aux = N_ig*Xe;
    f_ig = SourceTerm(aux);
    fe = fe + (N_ig+tau*(nu*(Nx+Ny) +(ax*Nx+ay*Ny)))'*(f_ig*dvolu + sgma);
end
```

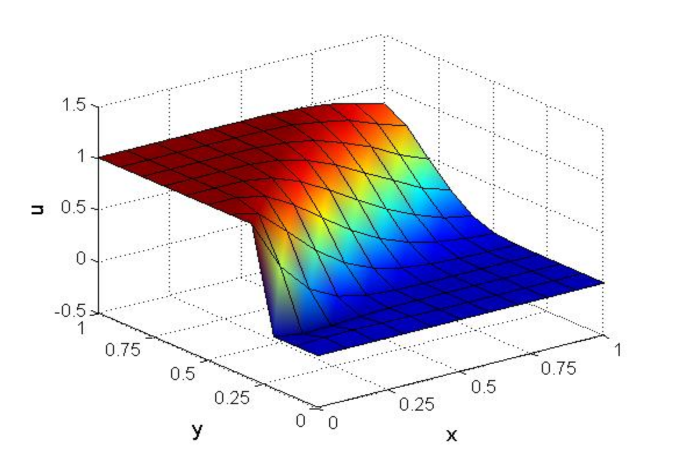
Fig 1: Implementation of GLS method

```
if BC==1
% nodes on which solution is u=1
nodesDir1 = nodes_x0( X(nodes_x0,2) > 0.2 );
% nodes on which solution is u=0
nodesDir0 = [nodes_x0( X(nodes_x0,2) <= 0.2 ); nodes_y0];
% Boundary condition matrix
C = [nodesDir1, ones(length(nodesDir1),1);
     nodesDir0, zeros(length(nodesDir0),1)];
elseif BC==2
% nodes on which solution is u=1
nodesDir1 = nodes_x0( X(nodes_x0,2) > 0.2 );
% nodes on which solution is u=0
nodesDir0 = [nodes_x0( X(nodes_x0,2) <= 0.2 ); nodes_y0; nodes_x1; nodes_y1];
% Boundary condition matrix
C = [nodesDir1, ones(length(nodesDir1),1);
     nodesDir0, zeros(length(nodesDir0),1)];
else
fprintf('Wrong Boundary Condition option');
end
```

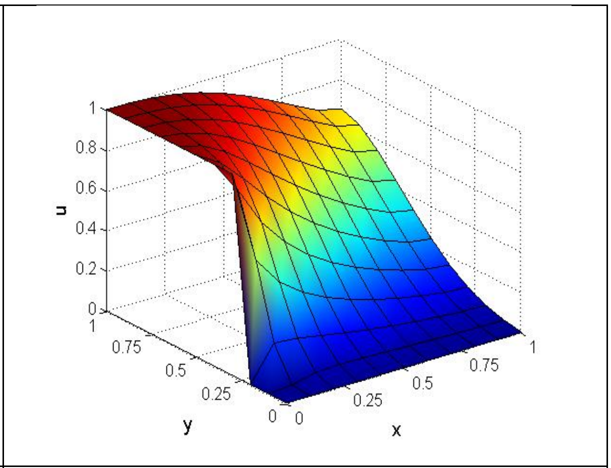
Fig 2: Modification in Boundary Conditions

Graphs:

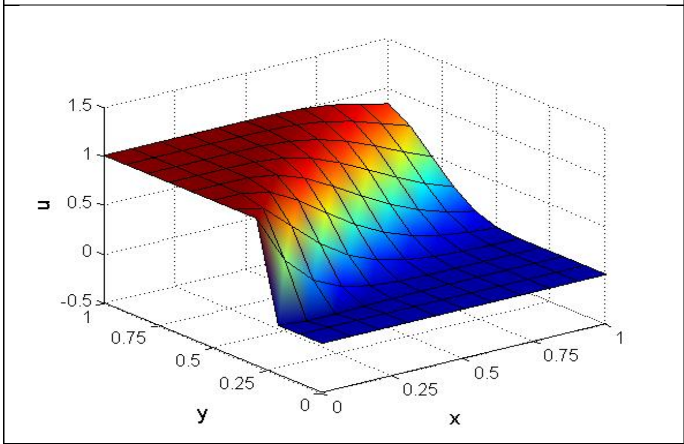
a. Neumann Boundary Condition (σ=0)



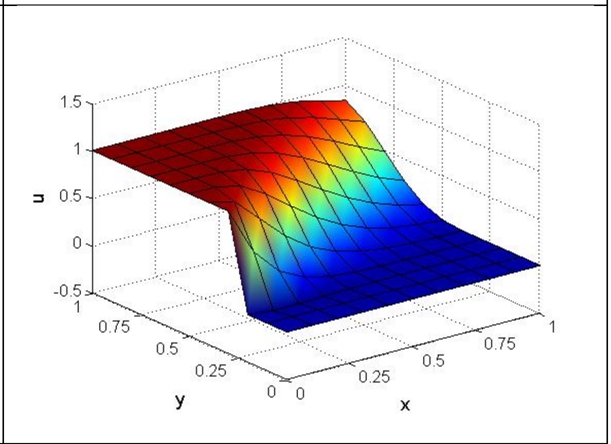
Galerkin



Artificial Diffusion

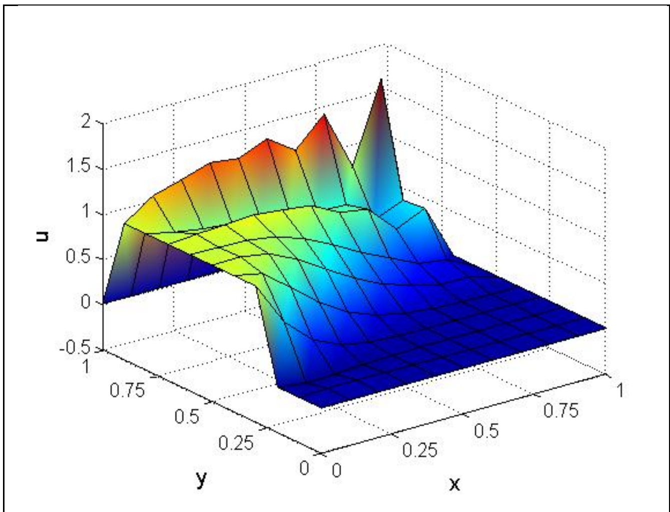


SPUG

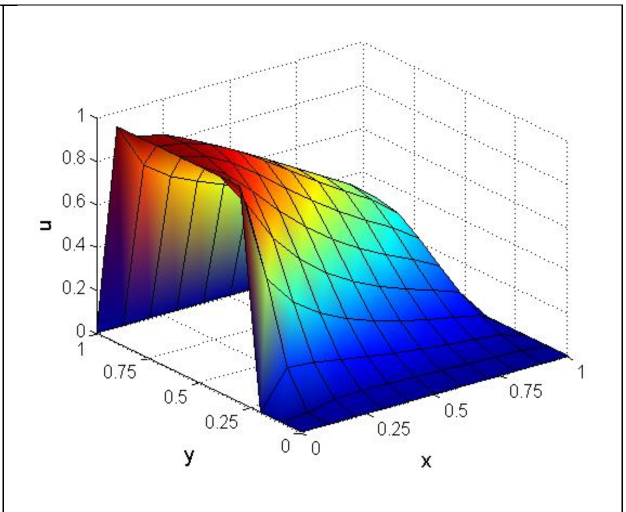


GLS

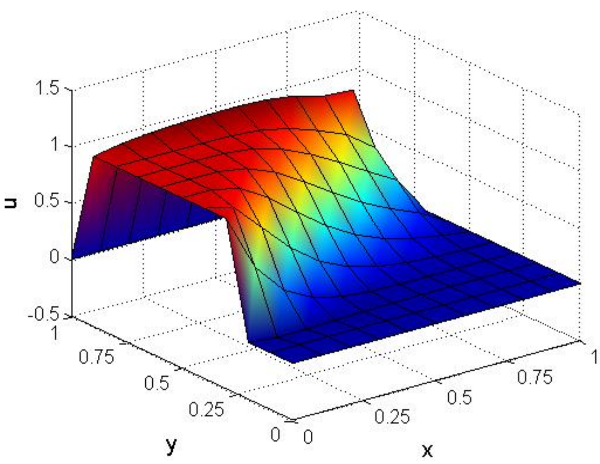
b. Dirichlet Boundary Conditions (σ=0) Galerkin Method produces spurious oscillations, shows instabilities while remaining methods show stability.



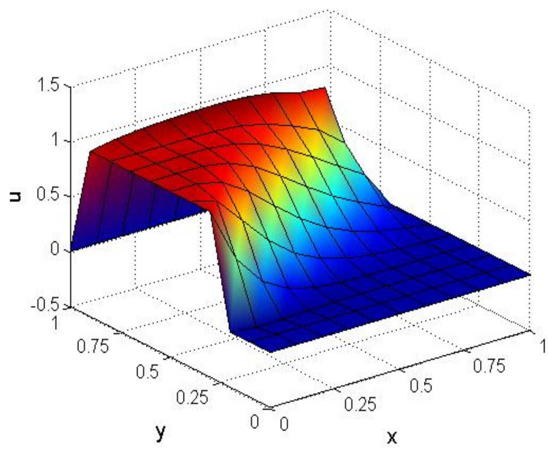
Galerkin



Artificial Diffusion

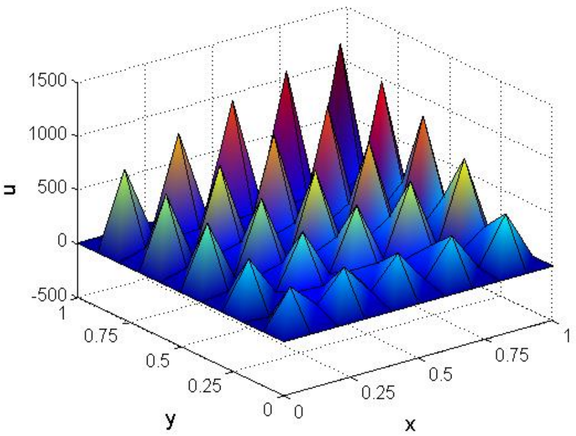


SUPG

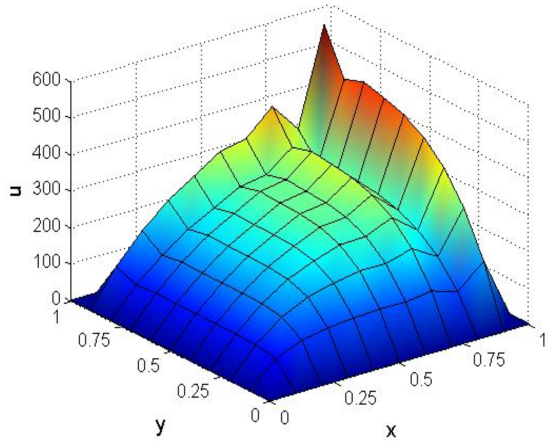


GLS

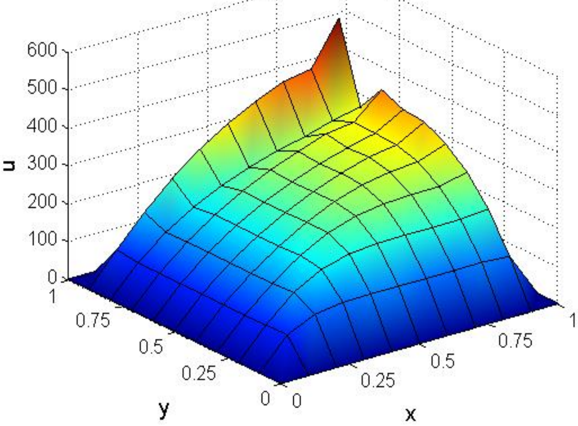
c. Convection-Reaction Dominated Case
Dirichlet Boundary Conditions (||*a*||=1/2, ν=10-4, σ=1), Galerkin method shows instabilities while all other methods are stable.



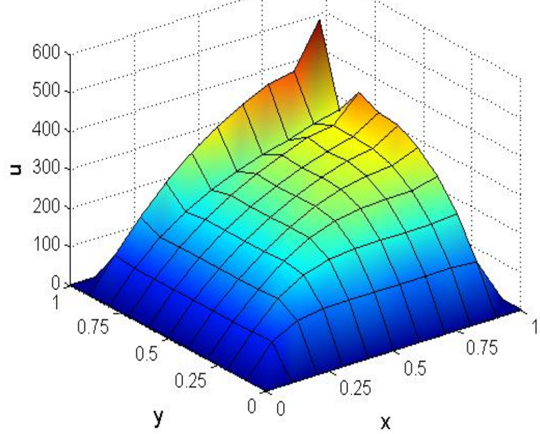
Galerkin



Artificial Diffusion

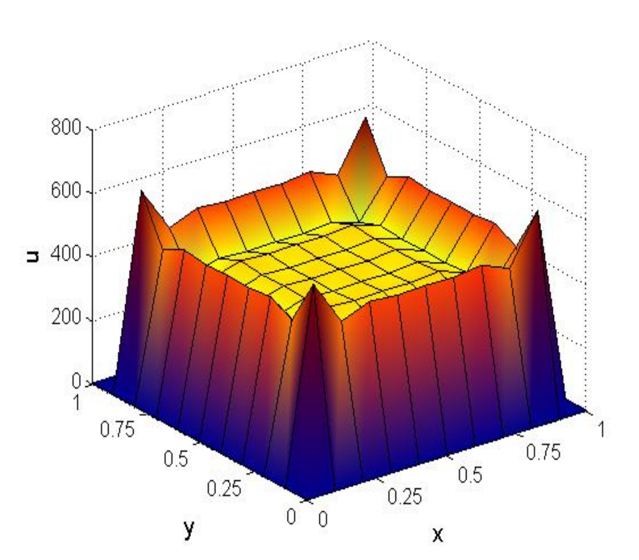


SUPG

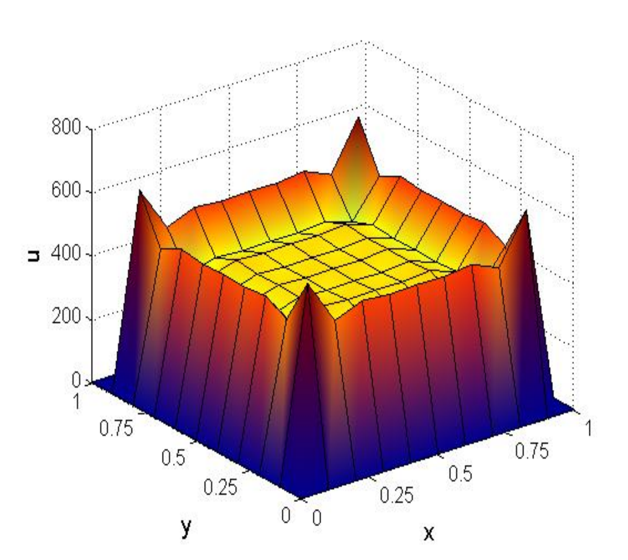


GLS

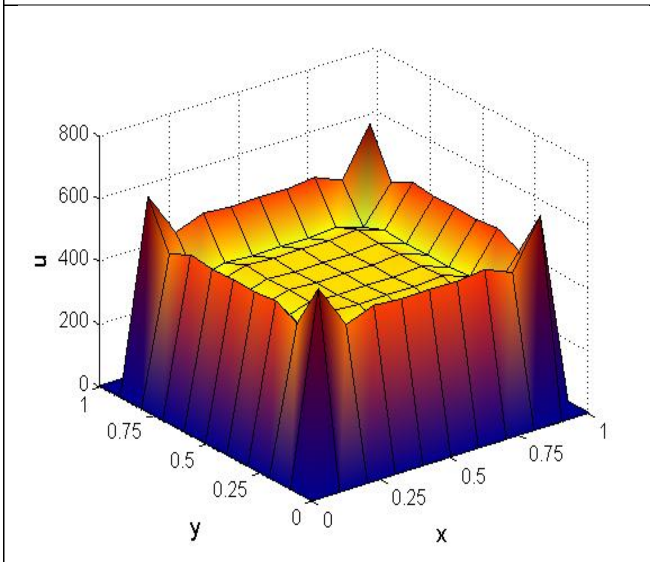
d. Reaction Dominated Case
   Dirichlet Boundary Conditions (||*a*||=10-3, ν=10-4, σ=1), all methods show stabilize results.



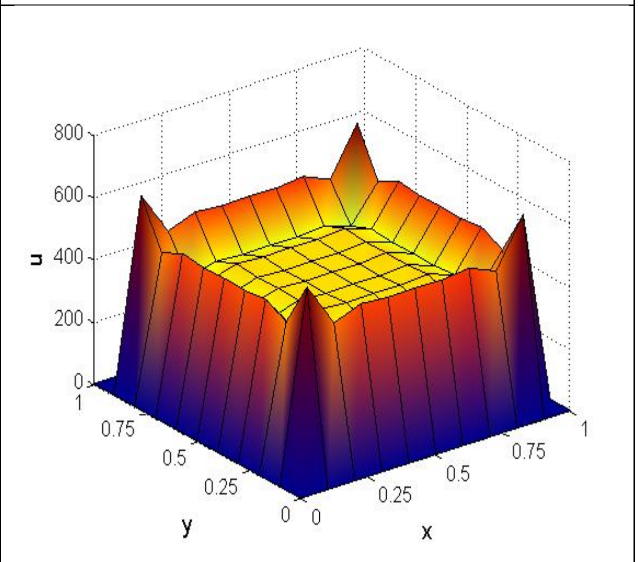
Galerkin



Artificial Diffusion



SPUG



GLS

B. 2D Unsteady Transport

Define Problem:

2D homogeneous convection equation with initial condition and homogeneous Dirichlet conditions on the inlet boundary

$$U(x,0) = 0.25 * (1 + \cos \pi X_1)(1 + \cos \pi X_2) \quad \text{If } X_1^2 + X_2^2 \leq 1$$
$$U(x,0) = 0$$

Introduction:

The numerical solution has been computed using following finite element schemes:

i. Lax-Wendroff + Galerkin (and with lumped mass matrix);
ii. Crank-Nicolson +Galerkin (and lumped mass matrix);
iii. The third-order explicit Taylor-Galerkin scheme (TG3);
iv. Two-step third order Taylor-Galerkin-2S method (TG3-2S);
v. Two-step fourth order Taylor-Galerkin method (TG4-2S).

Code:

The Matlab code for above mentioned methods was already given and some changes are made in the code for TG4-2S method. The changes are as follow,

```
elseif meth == 8
 %TG4-2S
    alpha = 1/12;
    A1 = M;
    B1 = -(dt/3)*C'- alpha*dt^2*(K - Co);
    f1 = (dt/3)*v1 + alpha*dt^2*(v2 - vo);
    A2 = M;
    B2 = -dt*C';
    C2 = - (dt^2/2)*(K-Co);
    f2 = dt*v1 - (dt^2/2)*(v2 - vo);
else
    error('Unavailable method')
end
```

Fig 1: Implementation of Two-Step Fourth order Taylor-Galerkin Method.

Term Representation in the Supplied Code:

a. Lax-Wendroff Method:

$$\frac{\Delta u}{\Delta t} = -a\nabla u^n + \frac{\Delta t}{2}(a.\nabla)^2 u^n + s^n + \frac{\Delta t}{2}(s_t^n - a.\nabla s^n)$$

weak form - ($s=0$ & $h=0$)

$$\left(w.\frac{\Delta u}{\Delta t}\right) = \left(a.\nabla w, u^n - \frac{\Delta t}{2}(a.\nabla)u^n\right) - \left((a.n)w, u^n - \frac{\Delta t}{2}(a.\nabla)u^n\right)_{\Gamma out}$$

terms representation - (supplied code)

$$A = M \longrightarrow (w.\Delta u)$$

$$B = \left(C - \left(\frac{\Delta t}{2}\right)K - Mo + \left(\frac{\Delta t}{2}\right)Co\right)\Delta t$$

ie,

$$C = (a.\nabla w, u^n)$$

$$K = (a.\nabla w)(a.\nabla u^n)$$

$$Mo = ((a.n)w, u^n)_{\Gamma out}$$

$$Co = ((a.n)w, (a.\nabla)u^n)_{\Gamma out}$$

b. Crank Nicolson Method:

$$\frac{\Delta u}{\Delta t} + \frac{1}{2}(a.\nabla)\Delta u = -a.\nabla u^n$$

weak form - ($s=0$ & $h=0$)

$$\left(w.\frac{\Delta u}{\Delta t}\right) - \frac{1}{2}(\nabla w, a\Delta u) + \frac{1}{2}((a.n)w.\Delta u)_{\Gamma out} = (\nabla w, a u^n) - ((a.n)w.u^n)_{\Gamma out}$$

terms representation (supplied code) -

(a) $A = m - \left(\frac{\Delta t}{2}\right)c + \left(\frac{\Delta t}{2}\right)mo$

where,

$$M = (w.\Delta u)$$

$$C = (\nabla w, a\Delta u)$$

$$Mo = ((a.n)w.\Delta u)_{\Gamma out}$$

(b) $B = c(\Delta t) - mo(\Delta t)$

where, $mo = ((a.n)w, u^n)_{\Gamma out}$.

c. Third Order Taylor Galerkin Method (TG3):

$$\left(1-\frac{\Delta t^2}{6}(a\cdot\nabla)^2\right)\frac{\Delta U}{\Delta t} = -a\cdot\nabla U^n + \frac{\Delta t}{2}(a\cdot\nabla)^2 U^n$$

weak form - (s = 0 & h = 0)

$$\left(w, \frac{\Delta U}{\Delta t}\right) + \frac{\Delta t^2}{6}\left(a\cdot\nabla w, \left(a\cdot\nabla \frac{\Delta U}{\Delta t}\right)\right) - \frac{\Delta t^2}{6}\left((a\cdot n)w, \left(a\cdot\nabla\frac{\Delta U}{\Delta t}\right)\right)_{\text{pout}} =$$

$$\left(a\cdot\nabla w, U^n - \frac{\Delta t}{2}(a\cdot\nabla)U^n\right) - \left((a\cdot n)w, U^n - \frac{\Delta t}{2}(a\cdot\nabla)U^n\right)_{\text{pout}}$$

terms representation (supplied code).

(a) $A = M + \left(\frac{\Delta t^2}{6}\right)(K - C_o)$

ie. $M = (w, \Delta U)$

$K = (a\cdot\nabla w)(a\cdot\nabla\Delta U)$

$C_o = ((a\cdot n)w, (a\cdot\nabla)\Delta U)_{\text{pout}}$

(b) $B = \left(C - \left(\frac{\Delta t}{2}\right)K - M_o + \left(\frac{\Delta t}{2}\right)C_o\right)\Delta t$

ie, $M_o = ((a\cdot n)w\cdot U^n)_{\text{pout}}$

d. Two Step Third Order Taylor Galerkin Method (TG3-2S):

weak form $[\alpha = \frac{1}{9}]$

$$\left(w, \frac{\bar{U}^n - U^n}{\Delta t}\right) = \frac{1}{3}(a\cdot\nabla w\cdot U^n) - \alpha(\Delta t)(a\cdot\nabla w, a\cdot\nabla U^n) + \alpha(\Delta t)((a\cdot n)w, \text{(a.} (a\cdot\nabla)U^n)_{\text{pout}}$$

$$\left(w, \frac{U^{n+1} - U^n}{\Delta t}\right) = (a\cdot\nabla w, U^n) - \frac{\Delta t}{2}(a\cdot\nabla w, a\nabla\bar{U}^n) + \frac{\Delta t}{2}((a\cdot n)w, (a\cdot\nabla)\bar{U}^n)_{\text{pout}}$$

Here, terms representation in 2 step,

$A_1 = M \longrightarrow M = (w, \bar{U}^n - U^n)$

$B_1 = -\left(\frac{\Delta t}{3}\right)C - \alpha(\Delta t^2)(K - C_o)$

where, $C = (a\cdot\nabla w, U^n)$

$K = (a\cdot\nabla w, a\nabla U^n)$

$A_2 = M \longrightarrow M = (w, U^{n+1} - U^n)$

$B_2 = -(\Delta t)C$

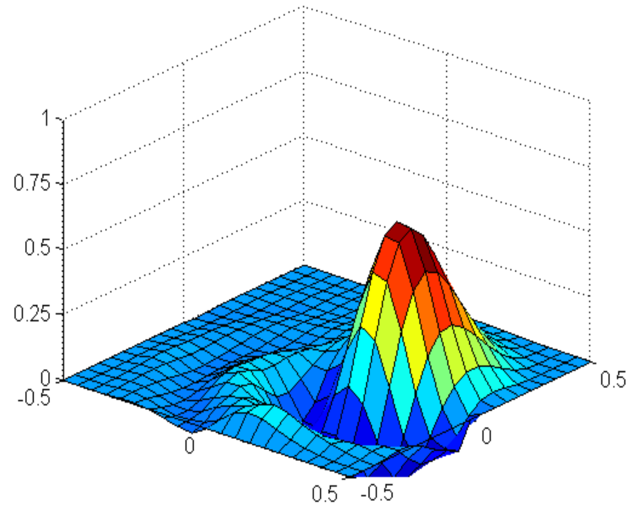
where, $C = (a\cdot\nabla w, U^n)$

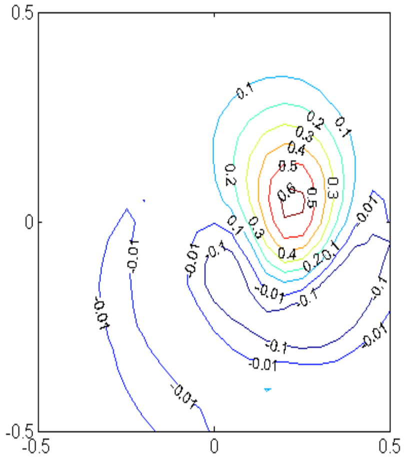
$C_2 = -\left(\frac{\Delta t^2}{2}\right)(K - C_o)$

where, $K = (a\cdot\nabla w, a\cdot\nabla\bar{U}^n)$

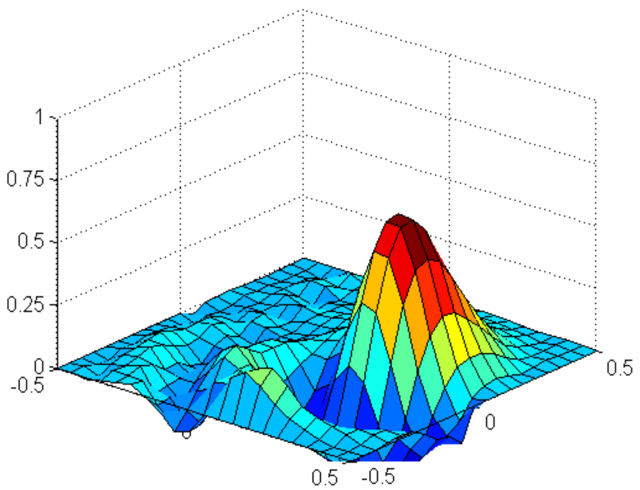
$C_o = ((a\cdot n)w, (a\cdot\nabla)\bar{U}^n)_{\text{pout}}$

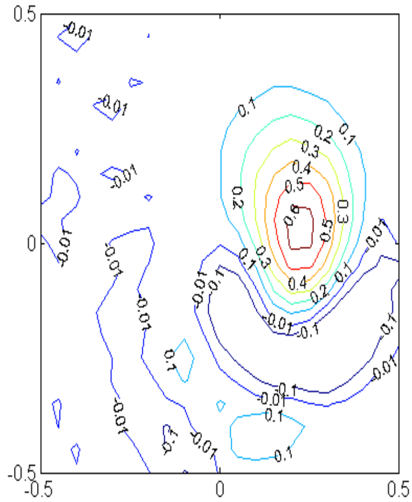
Graphs: Convection of cosine hill in pure rotation velocity field using finite element schemes.



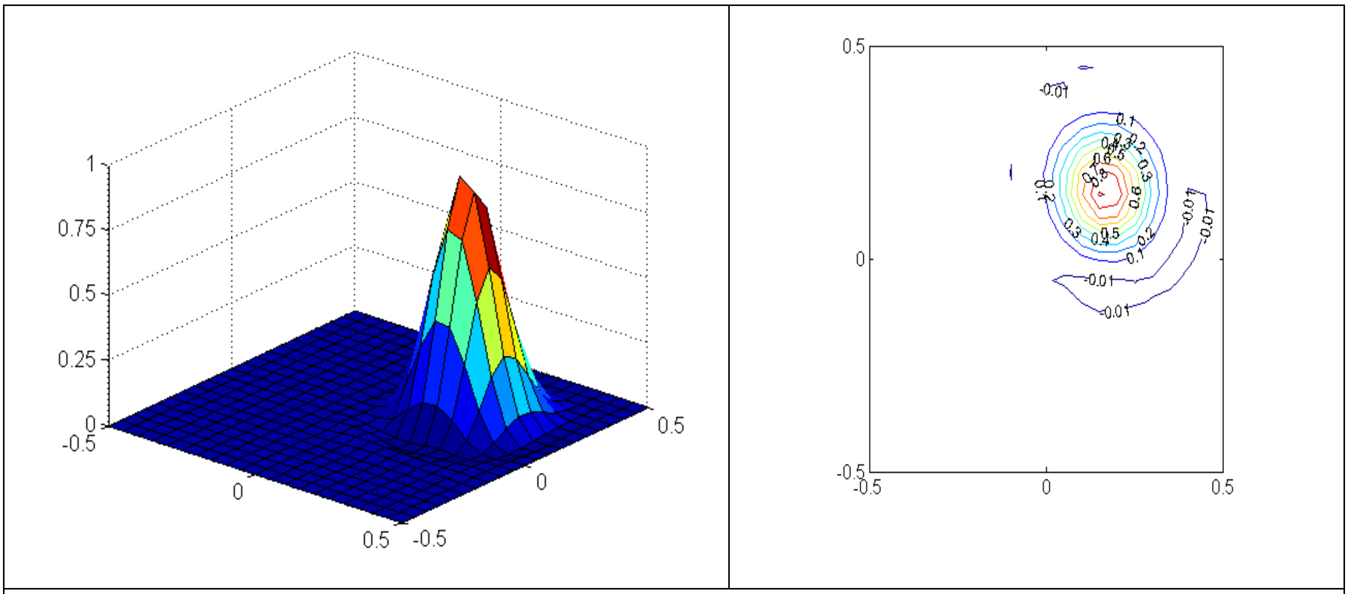


Lax Wendroff

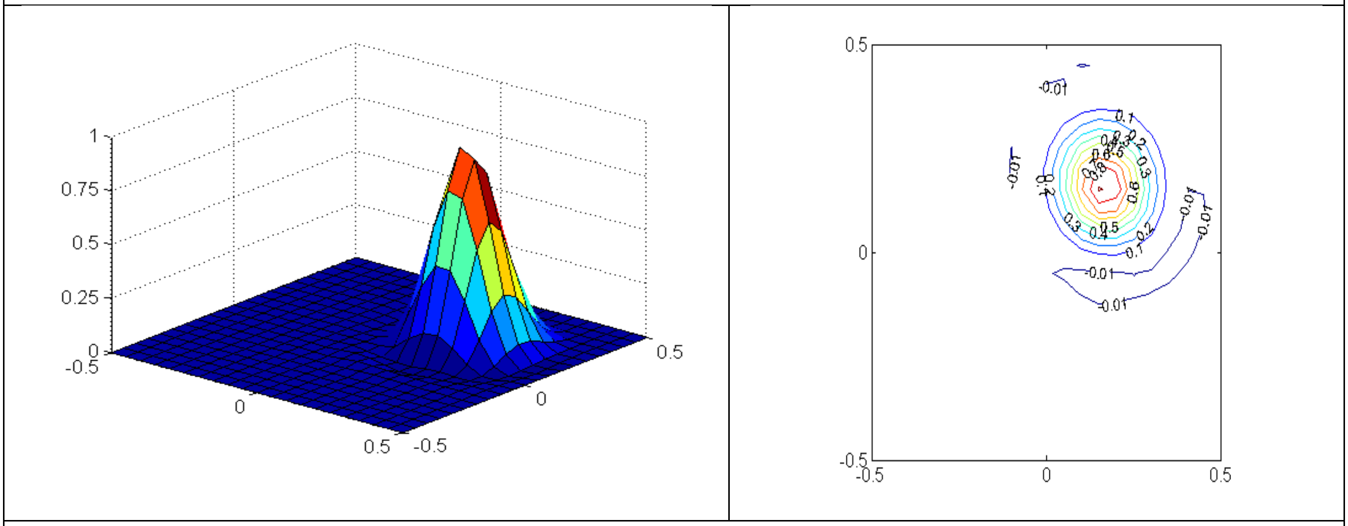




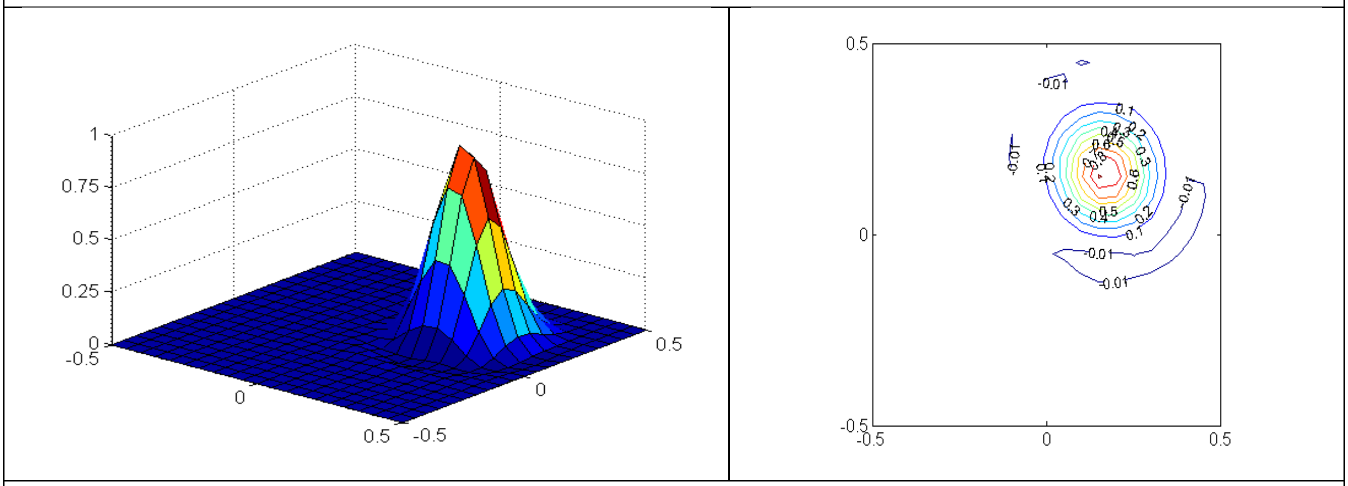
Crank Nicolson



TG3



TG3 2S



TG4 2S