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A coupling procedure for the convection-diffusion problem and the stokes problem

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1 Introduction

Many physical phenomena are governed by PDEs and these equations can be approximated by finite element (FE) methods, which results in a sparse linear system of equations to be solved via numerical linear algebra. The ever-increasing demand of reality in the simulation of the complex scientific and engineering problems faced nowadays involves the solution of coupled problems.

Coupled convection-diffusion and stokes problems are very common in industrial application. One classical application is in the transport of air contaminants. In some circumstances, the stokes problem that model the convective part of the problem is conditioned by the amount of contaminants and, in the other way around, contaminants are convected by the underlying (Navier-)Stokes problem. Another application of the coupled (Navier-)Stokes and convection-diffusion problems is in the medical industry which focuses on solving problems relevant to bioscience and biology. The proteins (actin filament in cells and hemoglobin in bloods) or monomers (amino acid and polysaccharides) exist in living body maintaining the vital signs. Here we are trying to model this kind of fluid with the goal to provide some numerical point of view for medical industry.

This report first provide the fractional-step method for solving Navier-Stokes equations, transient convection-diffusion equation numerically, the coupling procedure for the coupled problems, and finally show an example with the algorithm[1].

The original code is from the module "Finite Element in Fluids". In session of "unsteady-convection-diffusion equations" and session of "stokes flows", both problems are solved individually with linear parameters. Here the work is to couple the two kind of PDEs and also couple the existing code based on the developed algorithm.

2 Coupled convection-diffusion and stokes model

2.1 Physical background

We are going to propose a model which describes one kind of fluid which are incompressible and work as convection field for some substance, existing in many life entities.For example, in the computational biology community, there are many mathematical models trying to simulate the vital movement such as blood circulation and cell migration which play important roles in medical industry.

$$\begin{cases} \mathbf{u}_t - \nabla \cdot (\nu(\rho)\nabla)\mathbf{u} - \nabla p = 0 \quad with \\ \mathbf{u} = \mathbf{u}_D \quad on \quad \Gamma_D \\ \mathbf{u} = \mathbf{u}_N \quad on \quad \Gamma_N \end{cases} \tag{1}$$

$$\begin{cases} \rho_t + \mathbf{u} \cdot \nabla\rho - \nabla \cdot (\mu\nabla\rho) = s(\mathbf{u}) \quad with \\ \rho = \rho_D \quad on \quad \Gamma_D \\ \rho = \rho_N \quad on \quad \Gamma_N \end{cases} \tag{2}$$

This set of equations we are dealing with a fully coupled problem since the unknown depend on the solution of each other with set up a very broadly approach in engineering.It is a non-linear problem,so a linearization of the system have to be done.

2.2 Compact integral forms

In order to establish the weak, or variational, form of the strong form of our problem, we shall here make frequent application of the bilinear forms,

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} \colon \nabla\mathbf{v} d\Omega \quad \forall\mathbf{u},\mathbf{v}\in\mathcal{H}^1(\Omega) \tag{3}$$

$$b(\mathbf{v},q) = -\int_\Omega q\nabla\cdot\mathbf{v}d\Omega \quad \forall\mathbf{v}\in\mathcal{H}^1(\Omega) \quad q \quad q\in\mathcal{L}_2(\Omega) \tag{4}$$

and the trilinear form,

$$c(\mathbf{v};\mathbf{w},\mathbf{u}) = \int_\Omega \mathbf{w}\cdot(\mathbf{v}\cdot\nabla)\mathbf{u}d\Omega \quad \forall\mathbf{u},\mathbf{v},\mathbf{w}\in\mathcal{H}^1(\Omega) \tag{5}$$

2.3 Navier-Stokes equations

Consider the general form of Navier-Stokes equations prescribed a purely Dirichlet boundary condition:

$$\begin{cases} \mathbf{u}_t - \nu\nabla^2\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{b} \quad in \quad \Omega\times]0,T[\\ \nabla\cdot\mathbf{u} = 0 \quad in \quad \Omega\times]0,T[\\ \mathbf{u} = \mathbf{u}_D \quad on \quad \Gamma_D\times]0,T[\\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \quad in \quad \Omega \end{cases} \tag{6}$$

Solving equation 6 shows mainly two difficulties:

- Incompressibility constraint. The unknowns, velocity and pressure, cannot be discretized anyhow.Solution is guaranteed if interpolation spaces verify a stability condition known as inf-sup or LBB condition.
- Transient problem. The unknowns should advance in each time step which is similar to unsteady-convection-convection equation.

An effective method to solve the Navier-Stokes equations is the projection method computing the velocity field and pressure field separately by the computation of an intermediate velocity which is then projected onto the subspace of the solenoidal vector function.Basic to the derivation of the projection method is a theorem of orthogonal decomposition due to Ladyzhenskaya(1969),which is based on the Helmholtz decomposition principle.This theorem implies that any vector field $\mathbf{v}$ defined on a simply connected domain can be uniquely decomposed into a divergence-free(solenoidal) part $\mathbf{v}_{sol}$ and an irrotational part $\mathbf{v}_{irrot}$.Thus,

$$\mathbf{v} = \mathbf{v}_{sol} + \mathbf{v}_{irrot} = \mathbf{v}_{sol} + \nabla\phi \tag{7}$$

Since $\nabla\times\nabla\phi = 0$ for some scalar function,ϕ.Taking the divergence of equation 7,

$$\nabla\cdot\mathbf{v} = \nabla^2\phi \quad with \quad \nabla\cdot\mathbf{v}_{sol} = 0 \tag{8}$$

So we obtain a Poisson equation for the scalar function ϕ.If the vector field $\mathbf{v}$ is obtained first, the above equation 8 would help solve for the scalar function ϕ and the divergence-free part of $\mathbf{v}$ can extracted,

$$\mathbf{v}_{sol} = \mathbf{v} - \nabla\phi \tag{9}$$

Equation 9 is the key point and principle of solenoidal projection method for solving incompressible Navier-Stokes equations.

2.3.1 Fractional-step method for Navier-Stokes equations

Here we use the Chorin's projection method which is member of fractional methods. The typical algorithm for solving time-discretized equations of the projection method consists of two consecutive steps. First given previous time-step velocity field $\mathbf{u}^n$,an intermediate velocity field $\mathbf{u}_{int}^{n+1}$ is computed with the pressure term omitted,not satisfying the condition of incompressibility,shown as equation 10:

$$\begin{cases} \frac{\mathbf{u}_{int}^{n+1}-\mathbf{u}^n}{\triangle t} + (\mathbf{u}^* \cdot \nabla)\mathbf{u}^{**} - \nu\nabla^2\mathbf{u}^{**} = \mathbf{b}^{n+1} \\ \mathbf{u}_{int}^{n+1} = \mathbf{u}_D^{n+1} \quad on \quad \Gamma_D \end{cases} \tag{10}$$

for the treatment of the nonlinear convective term,there are three choices of velocities $\mathbf{v}^*$ and $\mathbf{v}^{**}$,

for the explicit Euler method,conditionally stable:

$$\mathbf{v}^* = \mathbf{v}^{**} = \mathbf{v}^n$$

for a semi-implicit method,unconditionally stable:

$$\mathbf{v}^* = \mathbf{v}^n \quad \mathbf{v}^{**} = \mathbf{v}_{int}^{n+1}$$

for the implicit Euler method, unconditionally stable:

$$\mathbf{v}^* = \mathbf{v}^{**} = \mathbf{v}_{int}^{n+1}$$

It is now necessary to derive a weak form of equation 10 to construct a finite element version for the first step.The goal for the current problem is to find the intermediate velocity $\mathbf{u}_{int}^{n+1} \in \mathcal{S}_{int}$, such that for all $\omega \in \mathcal{V}_{int}$,

$$(\mathbf{w}, \frac{\mathbf{u}_{int}^{n+1} - \mathbf{u}^n}{\triangle t}) + c(\mathbf{u}^*; \mathbf{w}, \mathbf{u}^{**}) + a(\mathbf{w}, \mathbf{u}^{**}) = (\mathbf{w}, \mathbf{b}^{n+1})$$

where the trilinear and bilinear forms have the same definition as before.And the functional spaces $\mathcal{S}_{int}$ and $\mathcal{V}_{int}$ completely fulfill the Dirichlet boundary condtions,namely $\mathbf{u}_{int}^{n+1} = \mathbf{u}_D^{n+1}$ on Γ.

For semi-implicit and fully implicit method, the discretized algebraic system resulting from Galerkin method is shown in equation 11

$$\mathbf{M}_1(\frac{\mathbf{u}_{int}^{n+1} - \mathbf{u}^n}{\triangle t}) + (\mathbf{C}(\mathbf{u}^*) + \mathbf{K})\mathbf{u}_{int}^{n+1} = \mathbf{f}^{n+1} \tag{11}$$

where $\mathbf{M}_1$ is the consistent mass matrix, $\mathbf{C}$ is the convection matrix, $\mathbf{K}$ is the viscosity matrix, and vector $\mathbf{f}^{n+1}$ includes the applied body force $\mathbf{b}$ and Dirichlet boundary conditions.

In terms of the computational complexity, the fully implicit option,$\mathbf{v}^* = \mathbf{v}_{int}^{n+1}$, requires the times integration to repeat computations of the inverse of the nonlinear and non-symmetric matrix $\mathbf{M}_1 + \triangle t(\mathbf{C}(\mathbf{u}_{int}^{n+1}) + K)$. Predictor-corrector methods are usually implemented in this case. For the semi-implicit methods,$\mathbf{v}^* = \mathbf{v}^n$,a modified convective term is adopted to maintain unconditional stability.

The *second step* of Chorin's projection is to determine the velocity $\mathbf{v}^{n+1}$ and pressure p^{n+1} solving equation 12

$$\begin{cases} \frac{\mathbf{u}^{n+1}-\mathbf{u}_{int}^{n+1}}{\triangle t} + \nabla p^{n+1} = 0 \quad in \quad \Omega \\ \nabla \cdot \mathbf{u}^{n+1} = 0 \quad in \quad \Omega \\ \mathbf{n} \cdot \mathbf{u}^{n+1} = \mathbf{n} \cdot \mathbf{u}_D^{n+1} \quad on \quad \Gamma \end{cases} \tag{12}$$

Note that the last step involves the remaining term (pressure) and condition (incompressibility) of the Navier-Stokes equations.However, the boundary condition only prescribes the normal component of the velocity,not the tangential components.This is an key aspect of the method: the tangential components of the velocity cannot be controlled on the boundary according to Helmholtz decomposition principle,where a condition on the normal component can be prescribed only.In accordance with equation 9, the first of equation 12 can be rewritten as

$$\mathbf{u}^{n+1} = \mathbf{u}_{int}^{n+1} - \triangle t \nabla p^{n+1}$$

Now, the weak form of the second step equation 12 is: find the end-of-step velocity $\mathbf{u}^{n+1} \in \mathcal{S}$ and the pressure $p^{n+1} \in \mathcal{Q}$,such that, $\forall\ (\mathbf{w}, q) \in \mathcal{V} \times \mathcal{Q}$,

$$\begin{cases} (\mathbf{w}, \frac{\mathbf{u}^{n+1}-\mathbf{u}_{int}^{n+1}}{\triangle t}) + b(\mathbf{w}, p^{n+1}) = 0 \\ b(\mathbf{u}^{n+1}, q) = 0 \end{cases} \tag{13}$$

where the function space $\mathcal{S}$ and $\mathcal{V}$ verify the prescribed boundary condition, $\mathbf{n} \cdot \mathbf{u}^{n+1} = \mathbf{n} \cdot \mathbf{u}_D^{n+1}$ on Γ. Similarly, the discretized formulation of equation 13 is,

$$\begin{cases} \mathbf{M}_2(\frac{\mathbf{u}^{n+1}-\mathbf{u}_{int}^{n+1}}{\triangle t}) + \mathbf{G}\mathbf{p}^{n+1} = 0 \\ \mathbf{G}^T\mathbf{u}^{n+1} = 0 \end{cases} \tag{14}$$

or equivalently,

$$\begin{pmatrix} \mathbf{M}_2/\triangle t & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}^{n+1} \\ \mathbf{p}^{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{M_2}\mathbf{u}_{int}^{n+1}/\triangle t \\ \mathbf{0} \end{pmatrix} \tag{15}$$

2.3.2 Viscosity splitting fractional-step method

Although it is enough to solve to the Navier-Stokes equation through Chorin's projection method,the difficult regarding the imposition of Dirichlet boundary condition in the second step still exists.To alleviate this problem, Balsco,Codina and Huerta(1997;1998) introduced a viscosity splitting fractional-step method where the projection idea is avoided in the second step. They add a new diffusion term in the momentum equation in the second step,which thus loses its inviscid property for preventing control of the prescribed tangential component of the velocity at the boundary, as shown in equation 16:

$$\begin{cases} \frac{\mathbf{u}^{n+1}-\mathbf{u}_{int}^{n+1}}{\triangle t} - \nu\nabla^2(\mathbf{u}^{n+1} - \mathbf{u}_{int}^{n+1}) + \nabla p^{n+1} = 0 \quad in \quad \Omega \\ \nabla \cdot \mathbf{u}^{n+1} = 0 \quad in \quad \Omega \\ \mathbf{u}^{n+1} = \mathbf{u}_D^{n+1} \quad on \quad \Gamma \end{cases} \tag{16}$$

Combined with equation 10 of the first step in Chorin's projection method, this formulation allow us to impose the original Dirichlet boundary conditions directly in both step,although the LBB condition is required to make it solvable.

2.3.3 Spatial discretization

The weak formulation can be obtained by projection equation 6 onto a space of weighting function $\mathbf{w}_1 \in \mathcal{V}$ for the momentum equation and $q \in \mathcal{Q}$ for the incompressibility condition.The following variational problem is thus the result:given $\mathbf{b}, \mathbf{u}_D$ and $\mathbf{u}_0$, find $\mathbf{u}(\mathbf{x}, t) \in \mathcal{S} \times]0, T[$ and $p(\mathbf{x}, t) \in \mathcal{Q} \times]0, T[$,such that, $\forall (\mathbf{w}_1, q) \in \mathcal{V} \times \mathcal{Q}$,

$$\begin{cases} (\mathbf{w}_1, \mathbf{u}_t) + a(\mathbf{w}_1, \mathbf{u}) + c(\mathbf{u}; \mathbf{w}_1, \mathbf{u}) + b(\mathbf{w}_1, p) = 0 \\ b(\mathbf{u}, q) = 0 \end{cases} \tag{17}$$

In a matrix form

$$\begin{cases} \mathbf{M}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u} + \mathbf{G}p(t) = \mathbf{f}(t, \mathbf{v}(t)) \\ \mathbf{G}^{\mathbf{T}}\mathbf{u}(t) = \mathbf{h}(t) \end{cases} \tag{18}$$

Let us denote some node indexes with superscripts a, b and the space indexes with subscripts i,j.Let N_u^a be the standard shape function associated to the velocity node a and N_p^c the standard shape function associated to be pressure node c. In general, the velocity and pressure interpolation may be different.So previous matrices in equation 18 are:

$$\mathbf{M}_{ij}^{ab} = (\mathbf{N}_u^a, \mathbf{N}_u^b)\delta_{ij} \quad G_j^{cb} = -(\mathbf{N}_p^c, \partial_j \mathbf{N}_u^b)$$

$$\mathbf{K}_{ij}^{ab} = (\mathbf{N}_u^a, \mathbf{u}^{n+\alpha} \cdot \nabla \mathbf{N}_u^b)\delta_{ij} + \nu(\nabla \mathbf{N}_u^a, \nabla \mathbf{N}_u^b)\delta_{ij}$$

$$\mathbf{h}(t) = -(\mathbf{v}_d(t), \mathbf{N}_p) \quad \mathbf{f}(t, \mathbf{v}(t)) = -\mathbf{K}^{ab}\mathbf{v}_d(t)$$

where δ_{ij} is the Kronecker δ.

2.3.4 Time discretization

Here we have to solve equation 18, a transient-diffusion Navier-Stokes equation.The technique used for unsteady-transportation-diffusion equation which updates the solution in each time step can also be applied to the current case.However, we could adopt *Chorin-Temam projection method*, where we compute the velocity and pressure fields through the computation of an intermediate velocity, which is then projected onto the subspace of the solenoidal vector function.

The *first step* includes the viscous and convective terms in the Navier-Stokes equation. Luckly we do not have convective terms here,thereby making $\mathbf{C}(\mathbf{u}^*) = 0$ in equation 11 to satisfy the current problem,

$$\mathbf{M}_1\left(\frac{\mathbf{u}_{int}^{n+1} - \mathbf{u}^n}{\triangle t}\right) + \mathbf{K}\mathbf{u}_{int}^{n+1} = \mathbf{f}^{n+1} \tag{19}$$

where $u^{**} = u_{int}^{n+1}$ for implicit Euler method.Rewrite equation 19 in terms of computation, the complete matrix form of first step for equation 6 is,

$$(\mathbf{M}_1 + \triangle t\mathbf{K})\mathbf{u}_{int}^{n+1} = \mathbf{M_1}\mathbf{u}^n + \triangle t\mathbf{f}^{n+1} \tag{20}$$

The *second step* of the projection method determines the end-of-step velocity $\mathbf{u}^{n+1}$ and pressure p^{n+1}. Instead of the original projection method in the second step,here we adopt the *viscosity splitting fractional-step method* as equation 16, which helps alleviate the difficulties regards the imposition of Dirichlet boundary conditions in the original method.The problem now is to translate equation 16 from strong form to a final matrix form,

$$\begin{bmatrix} \frac{\mathbf{M_2}}{\triangle t}+\mathbf{K} & \mathbf{G} \\ \mathbf{G^T} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u^{n+1}} \\ p^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{\mathbf{M_2 u}_{int}^{n+1}}{\triangle t}+\mathbf{K}\mathbf{u}_{int}^{n+1} \\ \mathbf{0} \end{bmatrix} \tag{21}$$

The final projection method has been applied to construct the matrix formulation for equation 6 into two consecutive,equation 20 and 21.

2.4 Convection-diffusion equation

2.4.1 Time discretization

Finite difference schemes are the most common method for time discretization,for example, θ-family method,Runge-Kutta methods and Padé' approximation.Here for simplification,we adopt θ-family methods which is widely used for integrating 1st-order differential equations. This is a single step method,meaning the solution ρ^{n+1} of the problem at time $t^{n+1} = t^n + \triangle t$ is determined by that at time t^n:

$$\frac{\rho(t^{n+1}) - \rho(t^n)}{\triangle t} = \theta \rho_t(t^{n+1}) + (1-\theta)\rho_t(t^n) + \mathcal{O}((1/2-\theta)\triangle t, \triangle t^2)$$

or, neglecting the truncation errors,

$$\frac{\triangle \rho}{\triangle t} - \theta \triangle \rho_t = \rho_t^n \quad where \quad \triangle \rho = \rho^{n+1} - \rho^n \tag{22}$$

Replacing ρ_t in equation 32, the time-discretized scheme is

$$\frac{\triangle \rho}{\triangle t} + \theta[\mathbf{u}\cdot\nabla - \nabla\cdot(\mu\nabla)]\triangle\rho = \theta s^{n+1} + (1-\theta)s^n - [\mathbf{u}\cdot\nabla - \nabla\cdot(\mu\nabla)]\rho^n \tag{23}$$

Several methods are obtained with different values of the θ parameter.The solution is conditionally stable if $\theta < 1/2$, for example,the forward Euler method,$\theta = 0$.On the other hand, the solution is unconditionally stable if $\theta \geq 1/2$, for example, Backward Euler,$\theta = 1$,Galerkin,$\theta = 2/3$, and Crank-Nicolson,$\theta = 1/2$ are the most usual ones.In these methods, Crank-Nicolson is the only method with second-order accuracy.

2.4.2 Spatial discretization

Sequentially, we implement the spatial discretization using Galerkin method, integrating over the computational domain and imposing the boundary conditions, we get equation 24

$$\begin{aligned} (w_2, \frac{\triangle \rho}{\triangle t}) + \theta[c(a; w_2, \triangle\rho) + a(w_2, \triangle\rho)] = \\ -[c(a; w_2, \rho^n) + a(w_2, \rho^n)] + \\ (w_2, \theta s^{n+1} + (1-\theta)s^n) + (w_2, \theta h^{n+1} + (1-\theta)h^n)_{\Gamma_n} \end{aligned} \tag{24}$$

with

$$a(w_2, \rho) = \int_\Omega \nabla w_2 \cdot (\mu \nabla \rho) d\Omega \quad (w_2, s) = \int_\Omega ws d\Omega$$

$$c(a; w_2, \rho) = \int_\Omega w_2 (a \cdot \nabla \rho) d\Omega \quad (w_2, h)_{\Gamma_n} = \int_{\Gamma_n} wh d\Gamma$$

If we consider the homogeneous Neumann condition, the discretized form can written as equation 25

$$[\frac{\mathbf{M}_3}{\triangle t} + \theta(\mathbf{C}_3 + \mathbf{K}_3)]\triangle\rho = -(\mathbf{C}_3 + \mathbf{K}_3)\rho^n + \mathbf{f}_3 \tag{25}$$

where

$$\mathbf{M}_3 = \int_\Omega N_i N_j d\Omega \quad \mathbf{C}_3 = \int_\Omega N_i(a\cdot\nabla N_j)d\Omega$$

$$\mathbf{K}_3 = \int_\Omega \nabla N_i\cdot(\mu\nabla N_j)d\Omega \quad \mathbf{f}_3 = \int_\Omega N_i(\theta s^{n+1} + (1-\theta)s^n)d\Omega$$

However, equation 25 does not consider the stabilization technique.Among many stabilization techniques such as Streamline-Upwind Petrov-Galerkin(SUPG) and the Galerkin/Least-squares methods. Here we choose the GLS for spatial discretization.Noting that here the residual is defined

$$R(\triangle\rho) := \frac{\triangle\rho}{\triangle t} - \theta\triangle\rho_t - \rho_t^n \tag{26}$$

And also, the perturbation operator $\mathcal{P}$ is

$$\mathcal{P} := \frac{w}{\triangle t} + \theta\mathcal{L}(w) \tag{27}$$

$$\begin{aligned}&(w_2, \frac{\triangle\rho}{\triangle t}) + \theta[c(a; w_2, \triangle\rho) + a(w_2, \triangle\rho)] + \tau(\frac{w}{\triangle t} + \theta\mathcal{L}(w), \mathcal{R}(\triangle\rho)) = \\ &-[c(a; w_2, \rho^n) + a(w_2, \rho^n)] + (w_2, \theta s^{n+1} + (1-\theta)s^n) + (w_2, \theta h^{n+1} + (1-\theta)h^n)_{\Gamma_n}\end{aligned} \tag{28}$$

$\Longleftrightarrow$

$$\begin{aligned}&(w_2, \frac{\triangle\rho}{\triangle t}) + \theta[c(a; w_2, \triangle\rho) + a(w_2, \triangle\rho)] + \frac{\tau}{\triangle t}(w, \mathcal{R}(\triangle\rho)) + \tau(\theta\mathcal{L}(w), \mathcal{R}(\triangle\rho)) = \\ &-[c(a; w_2, \rho^n) + a(w_2, \rho^n)] + (w_2, \theta s^{n+1} + (1-\theta)s^n) + (w_2, \theta h^{n+1} + (1-\theta)h^n)_{\Gamma_n}\end{aligned} \tag{29}$$

In a full matrix form,

$$[\frac{\mathbf{M}_3}{\triangle t} + \theta(\mathbf{C}_3 + \mathbf{K}_3) + \mathbf{A}_3]\triangle\rho = -(\mathbf{C}_3 + \mathbf{K}_3 + \mathbf{B}_3)\rho^n + \mathbf{f}_3 \tag{30}$$

where

$$\mathbf{A}_3 = \int_\Omega [\frac{N_i}{\triangle t} + \theta(\mathbf{u}\cdot\nabla N_i - \nabla\cdot\mu\nabla N_i)]\tau[\frac{N_j}{\triangle t} + \theta(\mathbf{u}\cdot\nabla - \nabla\cdot\mu\nabla)N_j]$$

$$\mathbf{f}_3 = \int_\Omega \{N_i + [\frac{N_i}{\triangle t} + \theta(\mathbf{u}\cdot\nabla N_i - \nabla\cdot\mu\nabla N_i)]\tau\}[\theta s^{n+1} + (1-\theta)s^n]d\Omega$$

$$\mathbf{B}_3 = \int_\Omega [\frac{N_i}{\triangle t} + \theta(\mathbf{u}\cdot\nabla N_i - \nabla\cdot\mu\nabla N_i)]\tau(\mathbf{u}\cdot\nabla N_j - \nabla\cdot\mu\nabla N_j)d\Omega$$

2.5 Coupling procedure

It is interesting to note that the original two equations are coupled by each other,where the solution of the Stokes equation is decided by the convection-diffusion equation,and vice versa.Precisely, the viscosity coefficient ν of Stokes equation is a function of ρ which is the solution from convection equation,and, the source term $s(\mathbf{u})$ is also a function of $\mathbf{u}$ which is obtained from the convection-diffusion equation. The two terms from the two equation establish the relationship between the equations.Although the matrix form of these equation have been derived, a clear solving procedure is also necessary,

step 1 Given initial conditions and boundary condition for equation 31 and equation 32.The initial conditions define the original states in the whole computational domain while the boundary conditions implies the solution on the boundary.

step 2 Define n as the total time step,i as the loop variable starting from 0. In this step, the total time is discretized into a consecutive small time step, and the loop variable clarifies a special time step.

step 3 Solve the convection-diffusion equation firstly to derive ρ^{i+1} with $\mathbf{u}^i$ and ρ^i in equation 30.Noting here the convective and source term are functions of $\mathbf{u}$,we have to update convection matrix and source vector in each time step.

step 4 Solve for $\mathbf{u}_{int}^{i+1}$ with ρ^{i+1} and equation 20. This is the first step of Chorin's projection method, computing an intermediate velocity field without the incompressibility condition.

step 5 Solve for $\mathbf{u}^{i+1}$ and p^{i+1} with $\mathbf{u}_{int}^{i+1}$ and equation21. A whole implementation of the projection method is finished after the determination of the end-of-step velocity and pressure.

step 6 After we have obtain the solution at a certain time step, we could update loop variable $i = i + 1$ meaning we are going to compute the velocity of next step through finite difference method.

step 7 Iterate from step 2 to step 7 until $i = n$.

3 Algorithm implementation and examples

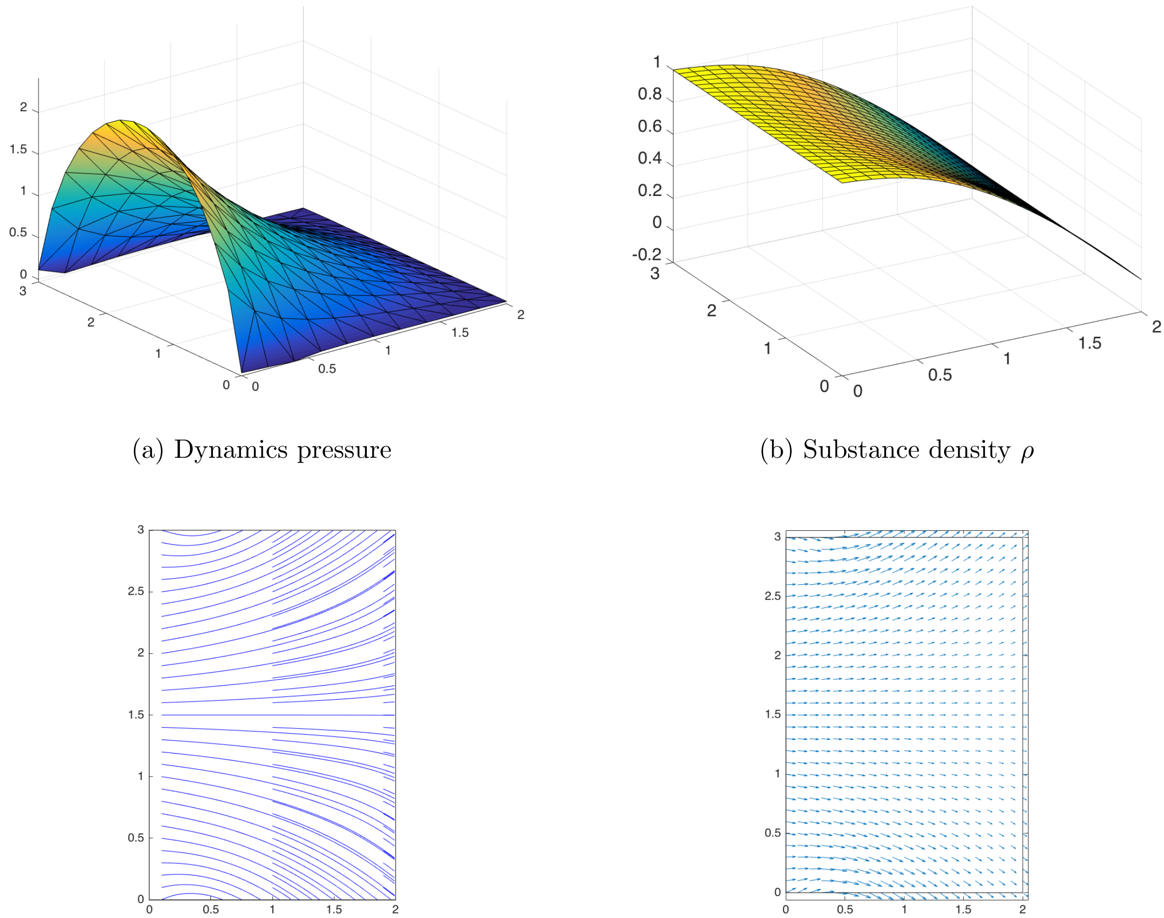
We solve the follow coupled problems as an examples for our algorithm:

$$\begin{cases} \mathbf{u}_t - \nabla \cdot (\nu \nabla)\mathbf{u} - \nabla p = 0 \quad with \\ v_x = 1 + sin(\omega t - \pi/2) \quad in \quad \Gamma_1, \Gamma_2 \end{cases} \tag{31}$$

$$\begin{cases} \rho_t + \mathbf{u} \cdot \nabla \rho - \nabla \cdot (\mu \nabla \rho) = s(\mathbf{u}) \quad with \\ \rho = 1 \quad in \quad \Gamma_1, \Gamma_2 \\ \rho = 0 \quad in \quad \Gamma_4 \end{cases} \tag{32}$$

where $\omega = 2\pi f$ and $f = 1/T$.$\mathbf{u}$ is the velocity of the Stokes flow, ρ is the density of a transported substance, μ is the diffusion coefficient of the substance and s($\mathbf{u}$) the source term.The viscosity component is given as $\nu = \nu_0 + \frac{\nu}{1+exp(-10(\rho-0.5))}$.Impose two values of ν_0 one that induce a convective dominant problem and other where the convective and diffusion effect are comparable. The source term is given as $s(\mathbf{u}) = \frac{1}{1+exp(-10(||\mathbf{u}||-0.5))}$.The reaction term will be $\sigma = 0$.

The code is developed based on the one from the *Finite element for flow problems* with some modifications. Figure 1 is the solution at the final time step, $t = 2 * \pi$. For the behavior in each time step, they are much like steady problem, symmetric field. Besides, another point,since the Dirichlet boundary condition for equation 32 is periodic, what we expect to see is the corresponding periodic dynamic behavior which could be seen from the code. Hereby the problem is solved.



(a) Dynamics pressure

(b) Substance density ρ

(c) Flow streamline

(d) Velocity field

Figure 1: Solution for transient coupled equation

4 Conclusions

In this project, we studied the projections method for solving general transient (Navier-)Stokes equations,and based on this method, the coupled algorithm for transient convection-diffusion equations and Stokes equations becomes easier to implemented in with time integration scheme. Results are well explained according to the boundary conditions and initial conditions. However, this work reported here also suggests that more work is needed to carry out systematic numerical experiment on more complicated initial conditions which would affect the behavior of the velocity field.

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6 Appendix

The code is available at my Github

References

[1] Jean Donea and Antonio Huerta. *Finite element methods for flow problems.* John Wiley & Sons, 2003.