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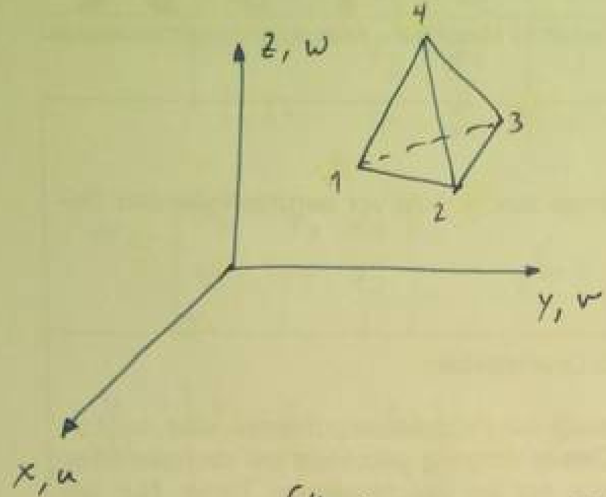
From Daryl L. Logan.

a First Course



$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$ unknown nodal displacement

3 dof/node × 4 node = 12 dof.

Displacement functions

$$u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z.$$

$$v(x,y,z) = a_5 + a_6 x + a_7 y + a_8 z$$

$$w(x,y,z) = a_9 + a_{10} x + a_{11} a_y + a_{12} z$$

then express the a_i's in terms of. know nodal coordinates $(x_1, y_1, z_1, \dots, z_4)$

$$u(x,y,z) = \frac{1}{6V}\{(\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 z)u_1$$
$$+ (\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 z)u_2.$$
$$+ (\alpha_3 + \beta_3 x + \gamma_3 y + \delta_3 z)u_3$$
$$+ (\alpha_4 + \beta_4 x + \gamma_4 y + \delta_4 z)u_4\}.$$

$$6V = \begin{array}{c|cccc|c} & \alpha & \beta & \gamma & \delta & \\ \hline & 1 & x_1 & y_1 & z_1 & 1 \\ & 1 & x_2 & y_2 & z_2 & 2 \\ & 1 & x_3 & y_3 & z_3 & 3 \\ & 1 & x_4 & y_4 & z_4 & 4 \end{array}$$

Where V is the volume of tetrahedron

$$\alpha_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \qquad \beta_1 = \begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}.$$

$$\gamma_1 = \begin{vmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix} \qquad \delta_1 = \begin{vmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & z_4 \end{vmatrix}$$

$$\alpha_2 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \qquad \beta_2 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}$$

$$_2 = \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix} \qquad \delta_2 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \\ 1 & x_4 & z_4 \end{vmatrix}$$

$$\alpha_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix} \qquad \beta_3 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_4 & z_4 \end{vmatrix}$$

$$\gamma_3 = \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_4 & z_4 \end{vmatrix} \qquad \delta_3 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_4 & z_4 \end{vmatrix}$$

$$\alpha_4 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad \beta_4 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}$$

$$\gamma_4 = \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{vmatrix} \qquad \delta_4 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

for $v(x,y,z) = \frac{1}{6V}\{(\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 z)v_1$

$+ (\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 z)v_2$

$+ (\alpha_3 + \beta_3 x + \gamma_3 y + \delta_3 z)v_3$

$+ (\alpha_4 + \beta_4 x + \gamma_4 y + \delta_4 z)v_4\}.$

for $w(x,y,z) = \frac{1}{6V}\{(\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 z)w_1$

$+ (\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 z)w_2$

$+ (\alpha_3 + \beta_3 x + \gamma_3 y + \delta_3 z)w_3$

$+ (\alpha_4 + \beta_4 x + \gamma_4 y + \delta_4 z)w_4\}.$

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \underbrace{\begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix}}_{\text{Shape function matrix}} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ u_3 \\ v_3 \\ w_3 \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$

$$N_1 = \frac{(\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 z)}{6V}$$

$$N_2 = \frac{(\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 z)}{6V}$$

$$N_3 = \frac{(\alpha_3 + \beta_3 x + \gamma_3 y + \delta_3 z)}{6V}$$

$$N_4 = \frac{(\alpha_4 + \beta_4 x + \gamma_4 y + \delta_4 z)}{6V}$$

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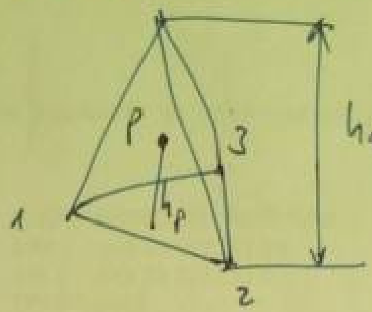
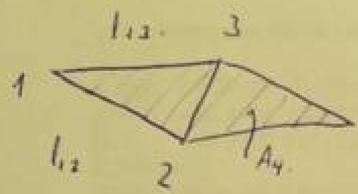
$F_P = p\, A_i$



$V = \frac{1}{3} A_i h_i$

$P(\xi_1, \xi_2, \xi_3, \xi_4) \rightarrow$ centroide.



$$l_{12} = \begin{Bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{Bmatrix}. \qquad l_{13} = \begin{Bmatrix} x_3 - x_1 \\ y_3 - y_1 \\ z_3 - z_1 \end{Bmatrix}.$$

$$|l_{12} \times l_{13}| = S_4 = 2A_4$$

$$l_{12} \times l_{13} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = \hat{i}\underbrace{\left[(y_2 - y_1)(z_3 - z_1) - (y_3 - y_1)(z_2 - z_1)\right]}_{l_x}.$$

$$-\hat{j}\underbrace{\left[(x_2 - x_1)(z_3 - z_1) - (x_3 - x_1)(z_2 - z_1)\right]}_{l_y}$$

$$+\hat{k}\underbrace{\left[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\right]}_{l_z}.$$

$$|l_{12} \times l_{13}| = \sqrt{l_x^2 + l_y^2 + l_z^2} = S_4 \rightarrow \text{Area of } 1,2,3$$

$$A_4 = \frac{S_4}{2}.$$

Surface forces.

$$\{f_s\} = \iint_S [N_s]^T \{T\}\, ds$$

$\{T\}$ → Traction vector components.

$[N_s]$ → Shape function matrix evaluated on the surface traction occurs.

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$$\{T\} = \{p\} = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

$$\{f_s\} = \iint_S [N]^T \Big|_{A_4} \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} ds$$

$A_4 \downarrow$ Surface 1,2,3

$$\{f_s\} = \frac{A_4}{3} \begin{Bmatrix} p_x \\ p_y \\ p_z \\ p_x \\ p_y \\ p_z \\ p_x \\ p_y \\ p_z \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

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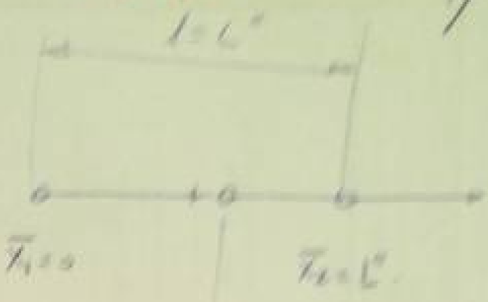

$$\begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{u}_1 & \bar{u}_2 & \bar{u}_3 \end{bmatrix} \begin{bmatrix} N_1^e(\xi) \\ N_2^e(\xi) \\ N_3^e(\xi) \end{bmatrix}$$

$$\sum_{i=1}^{3} N_i(\xi) = 1$$

$\bar{x}_3 = \frac{1}{2}L + \alpha L$

$= (\frac{1}{2} + \alpha)L$

$-\frac{1}{2} < \alpha < \frac{1}{2}$

$$\bar{x} = \sum_{i=1}^{3} \bar{x}_i N_i^e = [\bar{x}_1 \ \bar{x}_2 \ \bar{x}_3] \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

$$\bar{u} = \sum_{i=1}^{3} \bar{u}_i N_i^e = [\bar{u}_1 \ \bar{u}_2 \ \bar{u}_3] \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \end{bmatrix}$$

Local Coordinates (Isoparametric)

Isoparametric mapping.

$$N_1(\xi) = -\frac{\xi(1-\xi)}{2}$$

$$N_2(\xi) = \frac{\xi(1+\xi)}{2}$$

$$N_3(\xi) = 1 - \xi^2$$

$$\bar{x} = \sum_{i=1}^{3} N_i(\xi) \cdot x_i$$

$$\bar{x} = \frac{-\xi(1-\xi)}{2} x_1 + \frac{\xi(1+\xi)}{2} x_2 + (1-\xi^2) x_3.$$

$$\bar{x} = \frac{-\xi(1-\xi)}{2} \cdot 0 + \frac{\xi(1+\xi)}{2} \cdot l + (1-\xi^2)(\tfrac{1}{2}+\alpha) l$$

$$\bar{x} = \frac{\xi l}{2} + \frac{\xi^2}{2} l + \frac{1}{2} l + \alpha l - \frac{\xi^2}{2} l - \xi^2 \alpha l$$

$$\bar{x} = \underbrace{l(\tfrac{1}{2}+\alpha)}_{c} + \underbrace{\tfrac{l}{2}}_{b} \xi \underbrace{- \alpha l}_{a} \xi^2 \longrightarrow \bar{x}(\xi)$$

$$\bar{x} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Inverse mapping $\longrightarrow \xi(x)$.

$$\xi(\bar{x}) = \frac{-\frac{l}{2} \pm \sqrt{\frac{l^2}{4} - 4(-\alpha l)(l(\frac{1}{2}+\alpha))}}{2 \cdot (-\alpha l)} = \frac{\frac{l}{2} - \sqrt{\frac{l^2}{4} + 4\alpha l (l(\frac{1}{2}+\alpha))}}{2\alpha l}$$

$$= \frac{\frac{1}{2} - \sqrt{\frac{l^2}{4} + 4\alpha \frac{l^2}{2} + 4\alpha^2 l^2}}{2\alpha l} = \frac{\frac{1}{2} - \sqrt{\frac{1}{4} + 2\alpha + 4\alpha^2}}{2\alpha l}$$

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$$\xi(\bar{x}) = \frac{\frac{1}{2} - \sqrt{\frac{1}{4} + 2\alpha + 4\alpha^2}}{2\alpha}$$

$$\bar{x} = \sum_{i=1}^{3} \bar{x}_i N_i(\xi) = l\left(\frac{1}{2} + \alpha\right) + \frac{l}{2}\xi - \alpha l \xi^2 .$$

$$\frac{d\bar{x}}{d\xi} = \sum_{i=1}^{3} \frac{dN_i(\xi)}{d\xi} \bar{x}_i = J \quad (\text{Jacobian of mapping})$$

$$\frac{d\bar{x}}{d\xi} = \frac{l}{2} - 2\alpha l \xi = l\left(\frac{1}{2} - 2\alpha\xi\right).$$

$$\frac{\partial N_i^e}{\partial \bar{x}} = \frac{\partial N_i^e}{\partial \xi} \cdot \frac{\partial \xi}{\partial \bar{x}}$$

$$\frac{dN_1(\xi)}{d\xi} = -\frac{1}{2}\left((1-\xi) + \xi(-1)\right) = \frac{1}{2}(1 - 2\xi) = \xi - \frac{1}{2}$$

$$\frac{dN_2(\xi)}{d\xi} = \frac{1}{2}\left((1+\xi) + \xi\right) = \frac{1}{2}(1 + 2\xi) = \xi + \frac{1}{2}$$

$$\frac{dN_3(\xi)}{d\xi} = -2\xi$$

$$J = \frac{dN_1(\xi)}{d\xi}\bar{x}_1 + \frac{dN_2(\xi)}{d\xi}\bar{x}_2 + \frac{dN_3(\xi)}{d\xi}\bar{x}_3 .$$

$$J = \left(\xi - \frac{1}{2}\right)\cdot 0 + \left(\xi + \frac{1}{2}\right) l + (-2\xi)\left(\frac{1}{2} + \alpha\right) l$$

$$J = \xi l + \frac{1}{2} l - 2\xi \cdot \frac{1}{2} l - 2\xi\alpha l$$

$$J = \frac{1}{2} l - 2\xi\alpha l = l\left(\frac{1}{2} - 2\xi\alpha\right).$$

$$J = l\left(\frac{1}{2} - 2\xi\alpha\right).$$

if $\xi = 1$, then

$$\frac{1}{2} - 2\alpha = 0 \rightarrow J > 0$$

$$2\alpha = \frac{1}{2}$$

$$\alpha = \frac{1}{4} \rightarrow \frac{1}{3.9999}$$

if $\xi = -1$, then

$$\frac{1}{2} + 2\alpha = 0 \rightarrow J > 0$$

$$2\alpha = -\frac{1}{2}$$

$$\alpha = -\frac{1}{4} \rightarrow -\frac{1}{3.999}$$

$\therefore$ $\alpha = \pm\frac{1}{4}$ J vanishes

N_2^e:

$$N_2^e = C_2 \, L_{1-4} L_{3-4} L_{5-7} L_{6-8} = C_2(\xi+1)(\eta-1)\xi\eta$$

L_{1-4}: $\xi=-1 \rightarrow \xi+1=0 \rightarrow \xi+1$

L_{3-4}: $\eta=1 \rightarrow \eta-1=0 \rightarrow \eta-1$

L_{5-7}: $\xi=0 \rightarrow \xi$

L_{6-8}: $\eta=0 \rightarrow \eta$

for node 2: $\xi=1, \eta=-1 \rightarrow N_2^e(1,-1)=1$

$$N_2^e(1,-1) = C_2(1+1)(-1-1)(1)(-1) = C_2 \cdot 2 \cdot (-2) \cdot 1 \cdot (-1) = 4C_2 = 1$$

$$C_2 = \frac{1}{4}$$

$$\therefore \; N_2^e = \frac{1}{4}(\xi+1)(\eta-1)\xi\eta$$

N_3^e:

$$N_3^e = C_3 \, L_{1-2} L_{1-4} L_{5-7} L_{6-8} = C_3 \cdot (\eta+1)(\xi+1)\xi\eta \, .$$

L_{1-2}: $\eta=-1 \rightarrow \eta+1=0 \rightarrow \eta+1$

L_{1-4}: $\xi=-1 \rightarrow \xi+1=0 \rightarrow \xi+1$

L_{5-7}: $\xi=0 \rightarrow \xi$

L_{6-8}: $\eta=0 \rightarrow \eta$

for node 3: $\xi=1, \eta=1 \rightarrow N_3^e(1,1)=1$

$$N_3^e(1,1) = C_3(1+1)(1+1)1 \cdot 1 = C_3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = C_3 4 = 1$$

$$C_3 = \frac{1}{4}$$

$$\therefore \; N_3^e = \frac{1}{4}(\xi+1)(\eta+1)\xi\eta$$

N_4^e:

$$N_4^e = C_4 \, L_{1-2} L_{2-3} L_{5-7} L_{6-8} = C_4(\eta+1)(\xi-1)\xi\eta$$

L_{1-2}: $\eta=-1 \rightarrow \eta+1=0 \rightarrow \eta+1$

L_{2-3}: $\xi=1 \rightarrow \xi-1=0 \rightarrow \xi-1$

L_{5-7}: $\xi=0 \rightarrow \xi$

L_{6-8}: $\eta=0 \rightarrow \eta$

for node 4: $\xi=-1, \eta=1 \quad N_4^e(-1,1)=1$

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$$N_4^e(-1,1) = C_4\,(1+1)(-1-1)(-1)(1) = C_4 \cdot 2 \cdot (-2)(-1) \cdot 1 = 4C_4 = 1$$

$$C_4 = \frac{1}{4}$$

$$\therefore \quad N_4^e = \frac{1}{4}(\xi-1)(\eta+1)\xi\eta$$

N_5^e:

$$N_5^e = C_5\, L_{2-3}\, L_{1-4}\, L_{6-8}\, L_{3-4} = C_5\,(\xi-1)(\xi+1)\eta(\eta-1) =$$
$$= C_5(\xi^2-1)\eta(\eta-1)$$

L_{2-3}: $\xi = 1 \rightarrow \xi - 1 = 0 \rightarrow \xi - 1$

L_{1-4}: $\xi = -1 \rightarrow \xi + 1$

L_{6-8}: $\eta = 0 \rightarrow \eta$

L_{3-4}: $\eta = 1 \rightarrow \eta - 1$

for node 5 : $\xi = 0$; $\eta = -1$ $\quad N_5^e(0,-1) = 1$

$$N_5^e(0,-1) = C_5(0^2-1)(-1)(-1-1) = -2C_5 = 1$$

$$C_5 = -\frac{1}{2}$$

$$\therefore \quad N_5^e = -\frac{1}{2}(\xi^2-1)\eta(\eta-1)$$

N_6^e:

$$N_6^e = C_6\, L_{1-2}\, L_{3-4}\, L_{1-4}\, L_{5-7} = C_6\,(\eta+1)(\eta-1)(\xi+1)\xi$$
$$= C_6(\eta^2-1)(\xi+1)\xi$$

L_{1-2}: $\eta+1$

L_{3-4}: $\eta-1$

L_{1-4}: $\xi+1$

L_{5-7}: ξ

for node 6 : $\xi = 1$, $\eta = 0$ $\quad N_6^e(1,0) = 1$

$$N_6^e(1,0) = C_6(0^2-1)(1+1)\cdot 1 = -2C_6 = 1 \rightarrow C_6 = -\frac{1}{2}$$

$$\therefore \quad N_6^e = -\frac{1}{2}(\xi+1)(\eta^2-1)\xi$$

N_7^e:

$N_7^e = C_7\, L_{2-3}\, L_{1-4}\, L_{6-8}\, L_{1-2} = C_7\,(\xi-1)(\xi+1)\,\eta\,(\eta+1)$

$= C_7\,(\xi^2-1)\,\eta(\eta+1)$

L_{2-3}: $\xi - 1$

L_{1-4}: $\xi + 1$

L_{6-8}: η

L_{1-2}: $\eta + 1$

for node 7 : $\xi = 0$, $\eta = 1$ $\quad N_7^e(0,1) = 1$

$N_7^e(0,1) = C_7\,(0^2-1)\,1\cdot(1+1) = -2C_7 = 1 \longrightarrow C_7 = -\frac{1}{2}$

$\therefore \quad N_7^e = -\frac{1}{2}\,(\xi^2-1)\,\eta\,(\eta+1)$

N_8^e:

$N_8^e = C_8\, L_{1-2}\, L_{3-4}\, L_{5-7}\, L_{2-3} = C^8\,(\eta+1)(\eta-1)\,\xi\,(\xi-1)$

$= C^8\,(\eta^2-1)\,\xi(\xi-1)$

L_{1-2}: $\eta+1$

L_{3-4}: $\eta-1$

L_{5-7}: ξ

L_{2-3}: $\xi-1$

for node 8 : $\xi = -1$, $\eta = 0$ $\quad N_8^e(-1,0) = 1$

$N_8^e(-1,0) = C_8\,(0^2-1)(-1)(-1-1) = -2C_8 = 1 \longrightarrow C_8 = -\frac{1}{2}$

$\therefore \quad N_8^e = -\frac{1}{2}\,\xi(\xi-1)(\eta^2-1)$

N_9^e:

$N_9^e = C_9\, L_{1-2}\, L_{2-3}\, L_{3-4}\, L_{4-1} = C_9\,(\eta+1)(\xi-1)(\eta-1)(\xi+1)$

$= C_9\,(\eta^2-1)(\xi^2-1)$

for node 9 : $\xi = 0$, $\eta = 0$ $\quad N_9^e(0,0) = 1 = C_9\,(0^2-1)(0^2-1) = C_9 \rightarrow C_9 = 1$

$\therefore \quad N_9^e = (\xi^2-1)(\eta^2-1)$

$$x(\xi,\eta)=\left[\tfrac{1}{4}(\xi-1)(\eta-1)\eta\xi\right]\cancel{x_1}^{-1}+\left[\tfrac{1}{4}(\xi+1)(\eta-1)\xi\eta\right]\cancel{x_2}^{1}+\left[\tfrac{1}{4}(\xi+1)(\eta+1)\xi\eta\right]\cancel{x_3}^{1}+$$
$$+\left[\tfrac{1}{4}(\xi-1)(\eta+1)\xi\eta\right]\cancel{x_4}^{-1}+\left[-\tfrac{1}{2}(\xi^2-1)(\eta-1)\eta\right]\cancel{x_5}^{0}+\left[-\tfrac{1}{2}(\xi+1)(\eta^2-1)\xi\right]\cancel{x_6}^{1}+$$
$$+\left[-\tfrac{1}{2}(\xi^2-1)(\eta+1)\eta\right]\cancel{x_7}^{0}+\left[-\tfrac{1}{2}\xi(\xi-1)(\eta^2-1)\right]\cancel{x_8}^{-1}+\left[(\xi^2-1)(\eta^2-1)\right]\cancel{x_9}^{0}$$

$$x(\xi,\eta)=\frac{1}{4}\eta\xi\left[(\xi-1)(\eta-1)+(\xi+1)(\eta-1)+(\xi+1)(\eta+1)-(\xi-1)(\eta+1)\right]$$
$$-\frac{1}{2}\xi\left[(\xi+1)(\eta^2-1)+(\xi-1)(\eta^2-1)\right]$$

$$x(\xi,\eta)=\frac{1}{4}\eta\xi\left[(\eta-1)\left[\overbrace{(\xi+1)-(\xi-1)}^{2}\right]+(\eta+1)\left[\underbrace{(\xi+1)-(\xi-1)}_{2}\right]\right]$$
$$-\frac{1}{2}\xi(\eta^2-1)\left[(\xi+1)+(\xi-1)\right]$$

$$x(\xi,\eta)=\frac{1}{\cancel{4}_{2}}\eta\xi\cdot\cancel{2}\left[(\eta-1)+(\eta+1)\right]-\frac{1}{\cancel{2}}\xi(\eta^2-1)\cdot\cancel{2}\xi$$

$$x(\xi,\eta)=\frac{1}{\cancel{2}}\eta\xi\cdot\cancel{2}\eta-\xi^2(\eta^2-1)=\xi\eta^2-\xi^2(\eta^2-1)$$

$$x(\xi,\eta)=\xi\eta^2-\xi^2(\eta^2-1)=\xi\eta^2-\xi^2\eta^2+\xi^2$$

$$y(\xi,\eta)=\left[\tfrac{1}{4}(\xi-1)(\eta-1)\eta\xi\right]\cancel{y_1}^{-1}+\left[\tfrac{1}{4}(\xi+1)(\eta-1)\xi\eta\right]\cancel{y_2}^{-1}+\left[\tfrac{1}{4}(\xi+1)(\eta+1)\xi\eta\right]\cancel{y_3}^{1}+$$
$$+\left[\tfrac{1}{4}(\xi-1)(\eta+1)\xi\eta\right]\cancel{y_4}^{1}+\left[-\tfrac{1}{2}(\xi^2-1)(\eta-1)\eta\right]\cancel{y_5}^{-1}+\left[-\tfrac{1}{2}(\xi+1)(\eta^2-1)\xi\right]\cancel{y_6}^{0}+$$
$$+\left[-\tfrac{1}{2}(\xi^2-1)(\eta+1)\eta\right]\cancel{y_7}^{1}+\left[-\tfrac{1}{2}\xi(\xi-1)(\eta^2-1)\right]\cancel{y_8}^{0}+\left[(\xi^2-1)(\eta^2-1)\right]\cancel{y_9}^{0}$$

$$y(\xi,\eta)=\frac{1}{4}\eta\xi\left[-(\xi-1)(\eta-1)-(\xi+1)(\eta-1)+(\xi+1)(\eta+1)+(\xi-1)(\eta+1)\right]$$
$$-\frac{1}{2}\eta\left[-(\xi^2-1)(\eta-1)+(\xi^2-1)(\eta+1)\right]$$

$$y(\xi,\eta)=\frac{1}{4}\eta\xi\left[(\eta-1)\left[\overbrace{-(\xi-1)-(\xi+1)}^{-2\xi}\right]+(\eta+1)\left[\underbrace{(\xi+1)+(\xi-1)}_{2\xi}\right]\right]$$
$$-\frac{1}{2}\eta\left[(\eta-1)\left[\underbrace{-(\xi^2-1)+(\xi^2-1)}_{0}\right]\right]$$

$$y(\xi,\eta)=\frac{1}{4}\eta\xi\cdot2\xi\left[\underbrace{-(\eta-1)+(\eta+1)}_{2}\right]=\eta\xi^2$$

$$y(\xi,\eta)=\eta\xi^2$$

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$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \eta^2 - 2\xi\eta^2 + 2\xi & 2\eta\xi \\ 2\xi\eta - 2\xi^2\eta & \xi^2 \end{bmatrix}$$

$$\frac{\partial x}{\partial \xi} = \eta^2 - 2\xi\eta^2 + 2\xi \qquad \frac{\partial y}{\partial \xi} = 2\eta\xi$$

$$\frac{\partial x}{\partial \eta} = 2\xi\eta - 2\xi^2\eta \qquad \frac{\partial y}{\partial \eta} = \xi^2 .$$

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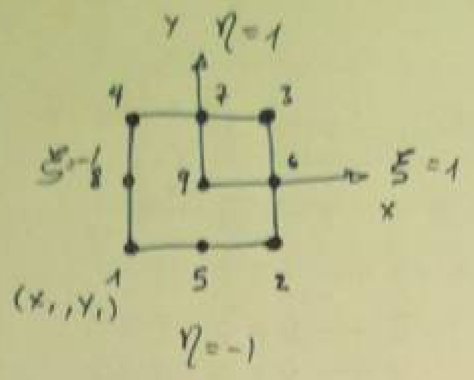

$-1 \leq \xi \leq 1$

$-1 \leq \eta \leq 1$

n	x	y
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0
9	0	0

$$\begin{bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 \\ u_{x_1} & u_{x_2} & u_{x_3} & u_{x_4} & u_{x_5} & u_{x_6} & u_{x_7} & u_{x_8} & u_{x_9} \\ u_{y_1} & u_{y_2} & u_{y_3} & u_{y_4} & u_{y_5} & u_{y_6} & u_{y_7} & u_{y_8} & u_{y_9} \end{bmatrix} \begin{bmatrix} N_1^e \\ N_2^e \\ N_3^e \\ N_4^e \\ \vdots \\ N_9^e \end{bmatrix}.$$

$$x = \sum_{i=1}^{9} x_i N_i^e = [x_1 \; x_2 \; x_3 \dots x_9] \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_9^e \end{bmatrix} \quad ; N_i^e = N_i^e(\xi, \eta)$$

$$y = \sum_{i=1}^{9} y_i N_i^e = [y_1 \; y_2 \; y_3 \; y_4 \; y_5 \dots y_9] \begin{bmatrix} N_1^e \\ N_2^e \\ \vdots \\ N_9^e \end{bmatrix} \qquad \sum_{i=1}^{9} N_i(\xi, \eta) = 1$$

Local coordinates

N_1^e:

$$N_1^e = C_1 \underbrace{L_{2-3} \cdot L_{3-4} \cdot L_{5-7} \cdot L_{6-8}}_{\text{interelement boundaries.}} = C_1 (\xi-1)(\eta-1)\eta\xi$$

L_{2-3}: $\xi = 1 \to \xi - 1 = 0 \to \xi - 1$

L_{3-4}: $\eta = 1 \to \eta - 1 = 0 \to \eta - 1$

L_{6-8}: $\eta = 0 \to \eta$

L_{5-7}: $\xi = 0 \to \xi$

for node 1 : $\xi = -1$, $\eta = -1 \to N_1^e(-1,-1) = 1$. : Normalization condition.

$$N_1^e(-1,-1) = C_1(-1-1)(-1-1)(-1)(-1) = C_1 \cdot (-2)(-2)(-1)(-1)$$
$$= C_1 4 = 1$$
$$C_1 = \frac{1}{4}$$

$\therefore$

$$N_1^e = \frac{1}{4}(\xi-1)(\eta-1)\eta\xi = \frac{1}{4}(\xi^2-\xi)(\eta^2-\eta)$$