ASSIGNMENT 1.
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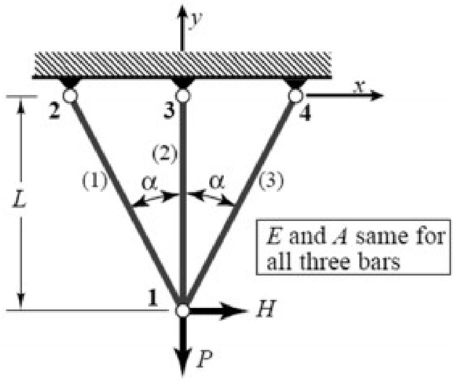
We have the truss problem with the geometric measures L and α angle and the applied forces shown in next figure:



It has 8 degrees of freedom, with 6 of them removable by the fixed displacement conditions at nodes 2, 3 and 4.

We follow the **direct stiffness method** to solve this problem.

Breakdown:
In the figure is shown the **idealization**, and the **loads and supports** are also shown.
We can think in this problem as three elements, also shown in figure as (1), (2) and (3). (**Disconnection and localization** steps).
Member element formation:
We solve each of these three elements separately.

For each element:

$\overline{u^e} = u^{e\prime} = T^e \, u^e$

$$\begin{pmatrix} u_{xi}' \\ u_{yi}' \\ u_{xj}' \\ u_{yj}' \end{pmatrix} = \begin{pmatrix} Cos\,\varphi & Sin\,\varphi & 0 & 0 \\ -Sin\,\varphi & Cos\,\varphi & 0 & 0 \\ 0 & 0 & Cos\,\varphi & Sin\,\varphi \\ 0 & 0 & -Sin\,\varphi & Cos\,\varphi \end{pmatrix} \begin{pmatrix} u_{xi} \\ u_{yi} \\ u_{xj} \\ u_{yj} \end{pmatrix}$$

$f^e = (T^e)^T f^{e\prime}$

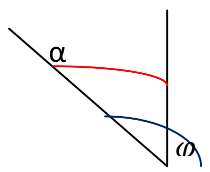
$$\begin{pmatrix} f_{xi} \\ f_{yi} \\ f_{xj} \\ f_{yj} \end{pmatrix} = \begin{pmatrix} Cos\,\varphi & -Sin\,\varphi & 0 & 0 \\ Sin\,\varphi & Cos\,\varphi & 0 & 0 \\ 0 & 0 & Cos\,\varphi & -Sin\,\varphi \\ 0 & 0 & Sin\,\varphi & Cos\,\varphi \end{pmatrix} \begin{pmatrix} f_{xi}' \\ f_{yi}' \\ f_{xj}' \\ f_{yj}' \end{pmatrix}$$

K^e=$(T^e)^T \; K^e \, T^e$

$E^1 = E^2$=E^3=E

$A^1 = A$=A^3=A

Element (1):



$\varphi = \frac{\pi}{2}$ + α

Cos φ = - Sin α = -s (for simplicity in the notation)

Sin φ = Cos α = c (for simplicity in the notation)

f_{x1} = $u_{x1} - u_{x2}$

f_{y1} = 0

f_{x2} = $-u_{x1} + u_{x2}$

f_{y2} = 0

So:

$$K^1 = \frac{E\,A}{L^1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

And we substitute Cos φ = -s and Sin φ = c in $(T^1)^T$ and (T^1) so we get:

$$K^1=(T^1)^T K^1 T^1 = \frac{E\,A}{L^1}\begin{pmatrix} -s & c & 0 & 0 \\ -c & -s & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -s & -c & 0 & 0 \\ c & -s & 0 & 0 \\ 0 & 0 & -s & -c \\ 0 & 0 & c & -s \end{pmatrix} =$$

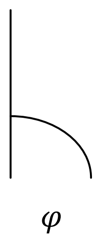
$$\frac{E\,A}{L^1}\begin{pmatrix} -s & 0 & s & 0 \\ -c & 0 & c & 0 \\ s & 0 & -s & 0 \\ c & 0 & -c & 0 \end{pmatrix}\begin{pmatrix} -s & -c & 0 & 0 \\ c & -s & 0 & 0 \\ 0 & 0 & -s & -c \\ 0 & 0 & c & -s \end{pmatrix}$$

And $L^1 = \frac{L}{c}$

Finally:

$$K^1 = \frac{E\,A}{L}\begin{pmatrix} s^2c & -sc^2 & -s^2c & sc^2 \\ -sc^2 & c^3 & sc^2 & -c^3 \\ -s^2c & sc^2 & s^2c & -sc^2 \\ sc^2 & -c^3 & -sc^2 & c^3 \end{pmatrix} \text{(s-> -s)}$$

Element (2):



Now
$\varphi = \frac{\pi}{2}$, so:
Cos φ = c =0 (for simplicity in the notation)
Sin φ = s= 1 (for simplicity in the notation)

f_{x1} =0
f_{y1} =$u_{y1} - u_{y3}$
f_{x3} = 0
f_{y3} = $-u_{y1} + u_{y3}$

So:

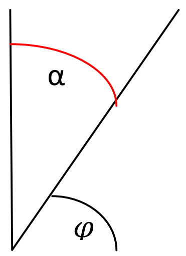
$$K^2 = \frac{E\,A}{L^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

And we substitute Cos φ = -s =0 and Sin φ = c=1 in $(T^1)^T$ and (T^1) so we get:

$$K^2 = (T^2)^T \; K^2 \, T^2 = \frac{E\,A}{L}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$L^2 = \frac{L}{c} = L$$

Element (3):



$\varphi = \frac{\pi}{2}$ - α

Cos φ = Sin α = s (for simplicity in the notation)

Sin φ = Cos α = c (for simplicity in the notation)

$f_{x1} = u_{x1} - u_{x4}$

$f_{y1} = 0$

$f_{x4} = -u_{x1} + u_{x4}$

$f_{y4} = 0$

So:

$$K^3 = \frac{E\,A}{L^1}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

And we substitute Cos φ = s and Sin φ = c in $(T^3)^T$ and (T^3) so we get:

$$K^3 = (T^3)^T\ K^3\ T^3 = \frac{E\,A}{L^1}\begin{pmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{pmatrix}$$

And $L^3 = \frac{L}{c}$

Finally:

$$K^3 = \frac{E\,A}{L^3}\begin{pmatrix} s & 0 & -s & 0 \\ -c & 0 & c & 0 \\ -s & 0 & s & 0 \\ c & 0 & -c & 0 \end{pmatrix}\begin{pmatrix} s & -c & 0 & 0 \\ c & s & 0 & 0 \\ 0 & 0 & s & -c \\ 0 & 0 & c & s \end{pmatrix} =$$

$$= \frac{E\,A}{L}\begin{pmatrix} s^2c & sc^2 & -s^2c & -sc^2 \\ sc^2 & c^3 & -sc^2 & -c^3 \\ -s^2c & -sc^2 & s^2c & sc^2 \\ -sc^2 & -c^3 & sc^2 & c^3 \end{pmatrix} \text{(s -> -s)}$$

Assembly:

element 1:

$$\begin{pmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{pmatrix} = \frac{E\,A}{L}\begin{pmatrix} s^2c & -sc^2 & -s^2c & sc^2 & 0 & 0 & 0 & 0 \\ -sc^2 & c^3 & sc^2 & -c^3 & 0 & 0 & 0 & 0 \\ -s^2c & sc^2 & s^2c & -sc^2 & 0 & 0 & 0 & 0 \\ sc^2 & -c^3 & -sc^2 & c^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{pmatrix}$$

element 2:

$$\begin{pmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{pmatrix} = \frac{E\,A}{L} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{pmatrix}$$

element 3:

$$\begin{pmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \\ f_{x3} \\ f_{y3} \\ f_{x4} \\ f_{y4} \end{pmatrix} = \frac{E\,A}{L} \begin{pmatrix} s^2c & sc^2 & 0 & 0 & 0 & 0 & -s^2c & -sc^2 \\ sc^2 & c^3 & 0 & 0 & 0 & 0 & s^2c & -c^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -s^2 & -sc^2 & 0 & 0 & 0 & 0 & s^2c & sc^2 \\ -sc^2 & -c^3 & 0 & 0 & 0 & 0 & sc^2 & c^3 \end{pmatrix} \begin{pmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{pmatrix}$$

$$f = f^{(1)} + f^{(2)} + f^{(3)} = \left(K^{(1)} + K^{(2)} + K^{(3)} \right) u = K\,u$$

$$\begin{pmatrix} H \\ -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{E\,A}{L} \begin{pmatrix} 2s^2c & 0 & -cs^2 & sc^2 & 0 & 0 & -cs^2 & -sc^2 \\ 0 & 1+2c^3 & sc^2 & -c^3 & 0 & -1 & -s^2c & -c^3 \\ -s^2c & sc^2 & s^2c & -c^2 & 0 & 0 & 0 & 0 \\ \dots & & & c^3 & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 & 0 \\ & & & & & 1 & 0 & 0 \\ & & & & & & -cs^2 & sc^2 \\ symm & & & & & & & c^3 \end{pmatrix} \begin{pmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ u_{x3} \\ u_{y3} \\ u_{x4} \\ u_{y4} \end{pmatrix}$$

The 5th row and column only contain zeros because they are related with the x component of the node 3, element 2, which has not any force acting in the x direction so no x direction displacement, just elongation in y direction.

Boundary conditions (BC):

According to the prescribed conditions, nodes 2, 3 and 4 are fixed, so:

$$\boldsymbol{u_{x2} = 0}$$
$$\boldsymbol{u_{y2} = 0}$$
$$\boldsymbol{u_{x3} = 0}$$
$$\boldsymbol{u_{y3} = 0}$$
$$\boldsymbol{u_{x4} = 0}$$
$$\boldsymbol{u_{y4} = 0}$$

And we obtain:

$$\frac{E\,A}{L}\begin{pmatrix} 2s^2c & 0 \\ 0 & 1+2c^3 \end{pmatrix}\begin{pmatrix} \boldsymbol{u_{x1}} \\ \boldsymbol{u_{y1}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{H} \\ \boldsymbol{-P} \end{pmatrix}$$

reducing the stiffness system 8x8 to a 2x2 system:

$$\frac{E\,A}{L} 2s^2c\, u_{x1} = H$$
$$\frac{E\,A}{L}(1+2c^3)u_{y1} = -P$$

So we can now solve the displacements u_{x1} and u_{y1}:

$$u_{x1} = \frac{H\,L}{2EAs^2c}$$

$$u_{y1} = \frac{-PL}{EA(1+2c^3)}$$

Let's check this solutions in the limit cases α → 0 and α → π/2

$$u_{x1} = \frac{H\,L}{2EAsi\ \ ^2\alpha \cos\alpha}$$

$$u_{y1} = \frac{-PL}{EA(1+2cos^2\alpha)}$$

When $\boldsymbol{\alpha} \to \boldsymbol{0}$: $(\cos\alpha \to 1, \sin\alpha \to 0)$

$u_{x1} \to \infty$

$$u_{y1} \rightarrow \frac{-PL}{3EA}$$

If $\alpha \rightarrow 0$ there is just one bar in the Y axis, so **no displacement can exist along x direction**. Just in y direction. So this has not physical sense.
This solution, with H≠0, physically can be interpreted such as a**ny force H (includes very small forces) applied in the x direction will cause an infinite displacement in X direction, which has no sense**.

When $\boldsymbol{\alpha \rightarrow \pi/2}$: $(\mathrm{Cos}\,\alpha \rightarrow 0, \mathrm{Sin}\,\alpha \rightarrow 1)$:

$$u_{x1} \rightarrow \infty$$
$$u_{y1} \rightarrow \frac{-PL}{EA}$$

This would be the case in which the distance from node 3 to 1 or 4 would be infinite. And node 1 will be in node 2. Now there is just one bar along the X axis, so **no displacement can exist along x direction**. Just in y direction. **So this has not physical sense**, again.

So this system has **not physical sense** for $\boldsymbol{\alpha \rightarrow 0 \text{ and } \alpha \rightarrow \pi/2}$:

For both cases should happens:

$u_{x1} \rightarrow 0$

And (H=0):

$u_{y1} \rightarrow \frac{-P}{3EA}$ (for $\alpha \rightarrow 0$)

$u_{y1} \rightarrow \frac{-PL}{EA}$ (for $\alpha \rightarrow \pi/2$)

<u>Axial forces</u>:

So we have an axial force for each of the 3 elements:

$$\overline{u_e} = T_e u_e$$

$$d_e = \bar{u}_{xj}{}^e - \bar{u}_{xi}{}^e$$

$$F_e = \frac{EA}{L} \cdot d_e$$

where d_e is the elongation and F_e then the axial force.

$$\overline{u_1} = \begin{pmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{pmatrix} \begin{pmatrix} \frac{H\,L}{2EAs^2c} \\ \frac{-PL}{EA(1+2c^3)} \\ 0 \\ 0 \end{pmatrix} = \left(\frac{H\,L}{2EAsc} + \frac{-PLc}{EA(1+2c^3)}, \frac{-H\,L}{2EAs^2} - \frac{PLs}{EA(1+2c^3)}, 0, 0\right)$$

$$d_1 = \bar{u}_{x2} \quad -\bar{u}_{x1} \quad = 0 - \frac{H\,L}{2EAsc} + \frac{PLc}{EA(1+2c^3)} = \frac{L}{EA}\left(\frac{-H}{2s} + \frac{Pc}{1+2c^3}\right)$$

$L^{(1)} = \frac{L}{c}$, L= $L^{(1)}c$

$$F^{(1)} = \frac{EA}{L} \cdot \frac{L}{EA}\left(\frac{-H}{2sc} + \frac{Pc}{1+2c^3}\right) = \frac{EA}{L^{(1)}} \cdot \frac{L^{(1)}c}{EA}\left(\frac{-H}{2sc} + \frac{Pc}{1+2c^3}\right) = \frac{-H}{2s} + \frac{Pc^2}{1+2c^3}$$

$$\overline{u_2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{H\,L}{2EAs^2c} \\ \frac{-PL}{EA(1+2c^3)} \\ 0 \\ 0 \end{pmatrix} = \left(\frac{-PL}{EA(1+2c^3)}, \frac{-H\,L}{2EAs^2c}, 0, 0\right)$$

$$d_2 = \bar{u}_{x2} \quad -\bar{u}_{x1} \quad = 0 - \frac{-PL}{EA(1+2c^3)}$$

$$L^{(2)} = L$$

$$F^{(2)} = \frac{EA}{L}\frac{PL}{EA(1+2c^3)} = \frac{P}{(1+2c^3)}$$

$L^{(3)} = \frac{L}{c}$, L= $L^{(3)}c$

$$\overline{u_3} = \begin{pmatrix} s & c & 0 & 0 \\ -c & s & 0 & 0 \\ 0 & 0 & s & c \\ 0 & 0 & -c & s \end{pmatrix} \begin{pmatrix} \frac{H\,L}{2EAs^2c} \\ \frac{-PL}{EA(1+2c^3)} \\ 0 \\ 0 \end{pmatrix} = \left(\frac{-PLc}{EA(1+2c^3)} + \frac{H\,L}{2EAsc}, \frac{-PLs}{EA(1+2c^3)} + \frac{H\,L}{2EAs^2}, 0, 0\right)$$

$$d_3 = \bar{u}_{x2} \quad -\bar{u}_{x1} \quad = 0 - \left(\frac{-PLc}{EAc(1+2c^3)} + \frac{H\,L}{2EAsc}\right) = -\frac{L^{(3)}c}{EA}\left(\frac{-Pc}{(1+2c^3)} + \frac{H}{2sc}\right)$$

$$F^{(3)} = \frac{EA}{L^{(3)}} \cdot (-\frac{L^{(3)}c}{EA}) \left(\frac{-Pc}{(1+2c^3)} + \frac{H}{2sc} \right) = \frac{P\ c^2}{(1+2c^3)} - \frac{H}{2s}$$

As we have seen $F^{(3)} = F^{(1)} = \frac{-H}{2s} + \frac{Pc^2}{1+2c^3}$

If H≠0 and α -> 0:

s-> 0 and c-> 1

So both $F^{(3)}\ and\ F^{(1)}$ -> ∞

This is because in this case we have just one bar and H is perpendicular to this bar, and we only would have P contribution. So H must be 0 if α=0 or α≠0 if H≠0, otherwise very small applied forces in the x axis, would produce infinite axial forces.