Polytechnic University of Catalonia

Msc. in Numerical Methods

Computational Structural Mechanics and Dynamics

Assignment 4:

"Iso-P Representation and Iso-P Quadrilateral Elements"

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4.1

- For $N_1(\xi) = a_0 + a_1(\xi) + a_2(\xi)^2$

$$N_1(0) = a_0 + a_1(0) + a_2(0)^2 = 0 \rightarrow a_0 = 0$$
$$N_1(1) = a_1(1) + a_2(1)^2 = 0 \rightarrow a_1 = -a_2$$
$$N_1(1) = a_1(-1) + a_2(-1)^2 = 1 \rightarrow -a_1 + a_2 = 1 \rightarrow 2a_2 = 1 \rightarrow a_2 = \frac{1}{2}$$
$$a_1 = -\frac{1}{2}$$

$$N_1(\xi) = -\frac{1}{2}(\xi) + \frac{1}{2}(\xi)^2 = \frac{1}{2}\xi(\xi - 1)$$

- For $N_2(\xi) = b_0 + b_1(\xi) + b_2(\xi)^2$

$$N_2(0) = b_0 + b_1(0) + b_2(0)^2 \rightarrow b_0 = 0$$
$$N_2(-1) = b_1(-1) + b_2(-1)^2 = 0 \rightarrow b_1 = b_2$$
$$N_2(1) = b_1(1) + b_2(1)^2 = 1 \rightarrow b_1 + b_2 = 1 \rightarrow 2b_1 = 1 \rightarrow b_1 = \frac{1}{2}$$
$$b_2 = \frac{1}{2}$$

$$N_2(\xi) = \frac{1}{2}(\xi) + \frac{1}{2}(\xi)^2 = \frac{1}{2}\xi(\xi + 1)$$

- For $N_3(\xi) = c_0 + c_1(\xi) + c_2(\xi)^2$

$$N_3(0) = c_0 + c_1(0) + c_2(0)^2 = 1 \rightarrow c_0 = 1$$
$$N_3(1) = 1 + c_1(1) + c_2(1)^2 = 0 \rightarrow c1_{=} - 1 - c_2$$
$$N_3(-1) = 1 + c_1(-1) + c_2(-1)^2 = 0 \rightarrow -c_1 + c_2 = -1 \rightarrow 2c_2 = -2 \rightarrow c_2 = -1$$
$$c_1 = 0$$

$$N_3(\xi) = 1 - \xi^2$$

The sum of the three functions is

$$N_1(\xi) + N_2(\xi) + N_3(\xi) = \cancel{\frac{1}{2}\xi^2} \cancel{-\frac{1}{2}\xi} \cancel{\frac{1}{2}\xi^2} \cancel{+\frac{1}{2}\xi} + 1 \cancel{-\xi^2} = 1$$

4.2

4.2.1.a

From the isoparametric definition of the element we know that:

$$x = x_1 N_1^e + x_2 N_2^e + x_3 N_3^e$$

Thus, the position of the nodes and the shape functions are known $x_1 = 0$, $x_2 = l$, $x_3 = (\frac{1}{2} + \alpha)l$ and $N_1(\xi) = \frac{1}{2}\xi(\xi - 1)$, $N_2(\xi) = \frac{1}{2}\xi(\xi + 1)$, $N_3(\xi) = 1 - \xi^2$

$$x = l\left(\frac{1}{2}\xi(\xi + 1)\right) + (\frac{1}{2} + \alpha)l\left(1 - \xi^2\right)$$
$$x = l(\frac{1}{2} + \alpha + \frac{1}{2}\xi - \alpha\xi^2)$$

The Jacobian for a 1D element is the derivative of the real coordinate with respect to the normalized one

$$J = \frac{dx}{d\xi} = l(\frac{1}{2} - 2\alpha\xi)$$

As l is always positive, then $(\frac{1}{2} - 2\alpha\xi) > 0$ in order to get $J > 0$. Additionally $-1 \leq \xi \leq 1$ as they are the limits of the normalized space. In order to get a positive Jacobian alpha should be bounded between $-\frac{1}{4}$ and $\frac{1}{4}$

4.2.1.b

By replacing $\alpha = 0$ in the Jacobian expression $J = \frac{dx}{d\xi} = \frac{l}{2}$. It is independent from the point in the normalized space where the Jacobian is evaluated, then it is constant all along the element.

4.2.2

The shape functions for the bar are:

$$N_1(\xi) = \frac{1}{2}\xi(\xi - 1)$$
$$N_2(\xi) = \frac{1}{2}\xi(\xi + 1)$$
$$N_3(\xi) = 1 - \xi^2$$

Differentiating with respect to ξ:

$$\frac{dN_1(\xi)}{d\xi} = \xi - \frac{1}{2}$$
$$\frac{dN_2(\xi)}{d\xi} = \xi + \frac{1}{2}$$
$$\frac{dN_3(\xi)}{d\xi} = -2\xi$$

The inverse of the Jacobian is:

$$J^{-1} = \frac{d\xi}{dx} = \frac{1}{l(0.5 - 2\alpha\xi)}$$

From the chain rule we can say that $B = \frac{dN}{dx} = \frac{dN}{d\xi}\frac{d\xi}{dx} = \frac{dN}{d\xi}J^{-1}$

$$B = \frac{1}{l(0.5 - 2\alpha\xi)} \begin{bmatrix} \xi - \frac{1}{2} \\ \xi + \frac{1}{2} \\ -2\xi \end{bmatrix}$$

4.2.3

The change of coordinates is given by the change of the limits of the integral and the integration parameter. As it is shown in figure 4.2 from the assignment the coordinates $x = 0$ and $x = l$ are transformed into $\xi = -1$ and $\xi = 1$ in the normalized space.

Also, $J = \frac{dx}{d\xi}$, and multiplying both sides by $d\xi$ we obtain $Jd\xi = \frac{dx}{d\xi}d\xi = dx$

4.3

The shape functions for a 5 nodded quadrilateral element, take the form of $N_i = N_i^{linear} - \alpha N_5$, where $\alpha = \frac{1}{4}$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)$$
$$N_2 = \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)$$
$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)$$
$$N_4 = \frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta^2)$$
$$N_5 = (1-\xi^2)(1-\eta^2)$$

As it can be observed in the following graphics, each shape function takes the value of 1 in it's respective node while 0 in the rest. Then, its evident that the sum of the five functions should be equal to 1 if evaluated in any nodal point.

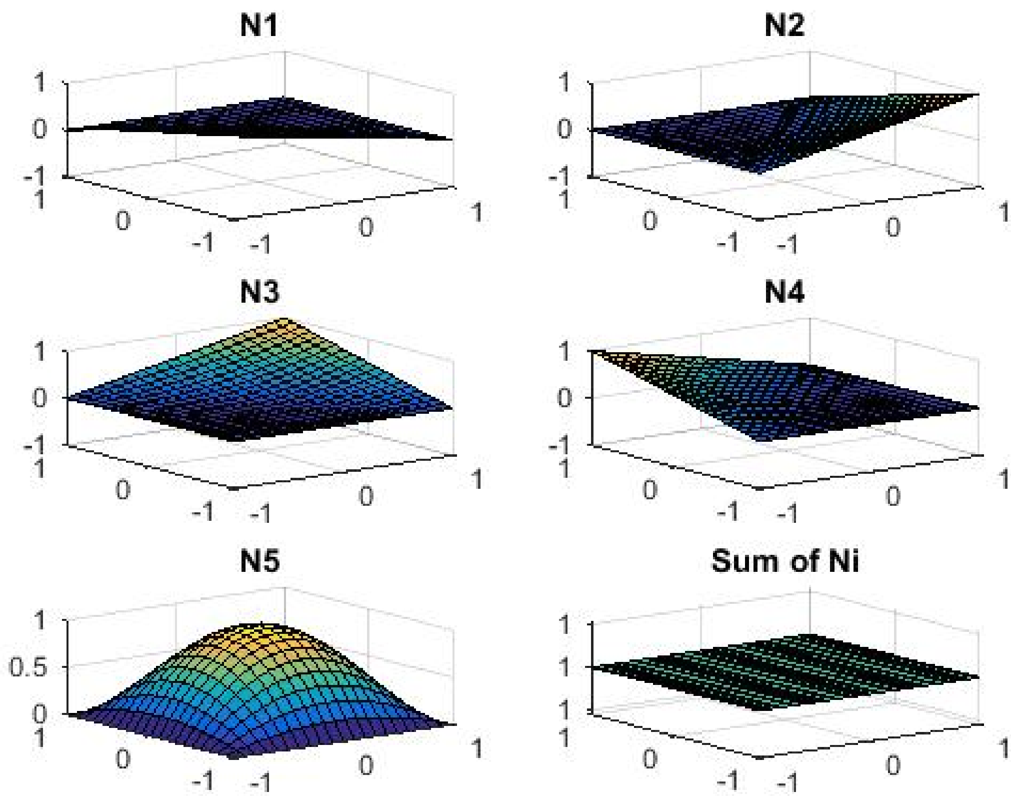

Figure: Shape functions